Shafts

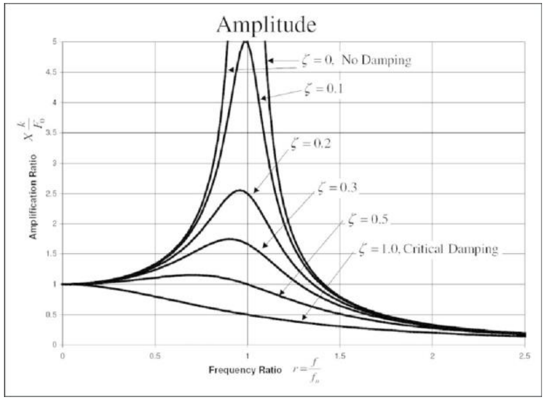
Critical Frequency

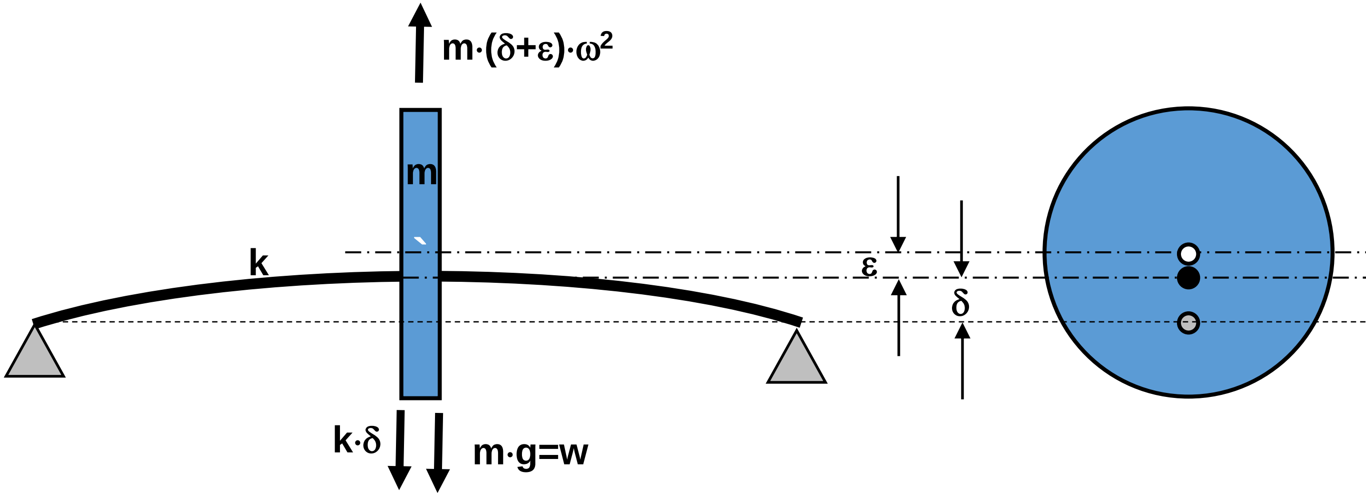
Vibration

- All systems containing energy storing elements will posses a set of natural frequencies
    - Moving masses store kinetic energy
    - Springs store potential energy

- Vibration occurs when there is a repeated transfer of potential to kinetic energy

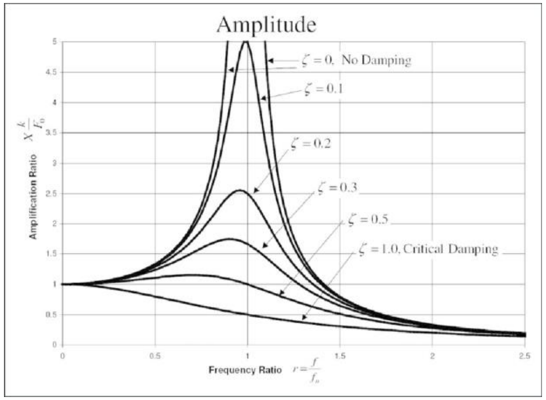
- Three types of Shaft Vibration
    - Lateral Vibration
    - Shaft Whirl
    - Torsional Vibration

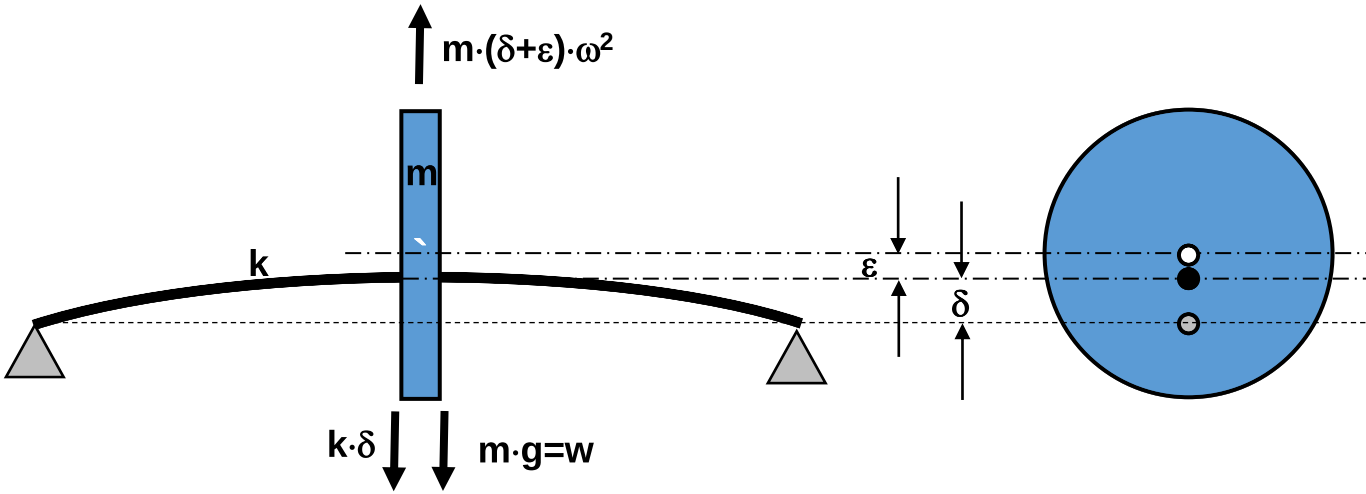




Vibration

- Shafts subjected to time-varying loads WILL vibrate
  - Free Vibration:
    - Subjected to transient load
    - Will eventually die out due to damping in the system
  - Force Vibration:
    - Time-varying load sustained
    - Time-varying load has a Forcing Frequency
    - Resonance occurs with Forcing Frequency coincides with the Natural Frequency
  - Natural Frequency → Critical Frequency → Critical Speed





Natural Frequency

- General Expression
  - Circular Frequency

$$\omega_n = \sqrt{\frac{k}{m}} \quad \left[{rad}/{s} \right]$$

  - Linear Frequency

$$f_n = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{k}{m}} \quad [Hz]$$

- Strategy
  - Keep all Forcing or Self-Exciting Frequencies below First Critical Frequency
  - The larger this margin the better, 3 to 4 times the best
  - Sometimes this is not possible, must accelerate quickly through natural frequency (Power Generation)

Nomenclature

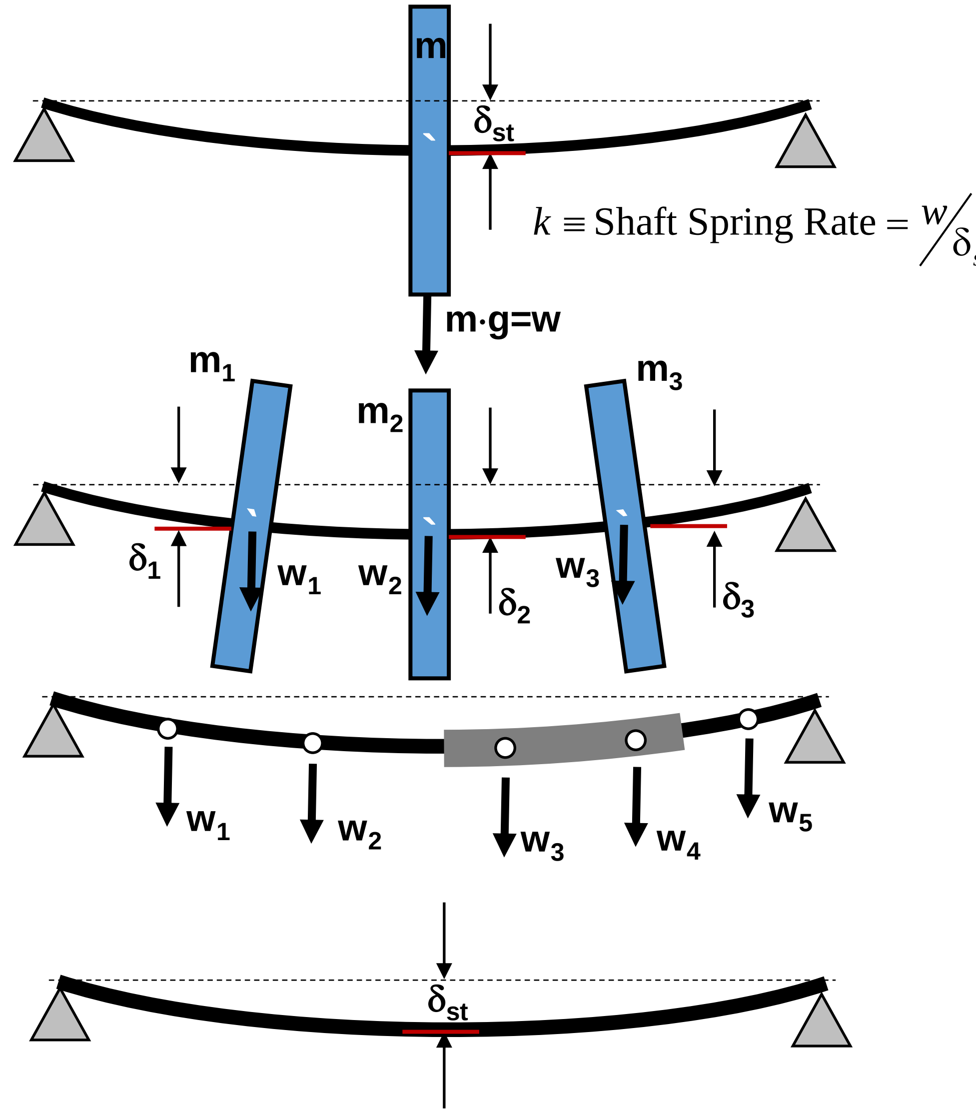
Critical Speed / Lowest Speed / Fundamental Speed

Quantity	Symbol	SI	Imperial
Mass	m	Kg	$lb \cdot s^2/ft$
Gravitational Force	w	N	lb
Static Deflection	δ_{st}	m	in
Shaft Spring Rate	k	N/m	lb/in
Acceleration of Gravity	g	m/s^2	in/s
Natural Frequency	ω_n	rad/s	rad/s
Critical Speed	n_c	rpm	rpm

Critical Speed Equations

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k \cdot g}{w}} = \sqrt{\frac{g}{\delta_{st}}}$$

$$n_c = \frac{30}{\pi} \cdot \sqrt{\frac{k}{m}} = \frac{30}{\pi} \cdot \sqrt{\frac{k \cdot g}{w}} = \frac{30}{\pi} \cdot \sqrt{\frac{g}{\delta_{st}}}$$



$$k \equiv \text{Shaft Spring Rate} = w/\delta_s$$

Rayleigh's Equation

$$\omega_n \approx \sqrt{\frac{g\left(w_1 \cdot \delta_1 + w_2 \cdot \delta_2 + \cdots\right)}{w_1 \cdot \delta_1^2 + w_2 \cdot \delta_2^2 + \cdots}}$$

$$n_c \approx \frac{30}{\pi} \cdot \sqrt{\frac{g\left(w_1 \cdot \delta_1 + w_2 \cdot \delta_2 + \cdots\right)}{w_1 \cdot \delta_1^2 + w_2 \cdot \delta_2^2 + \cdots}}$$

$$n_c \approx \frac{30}{\pi} \cdot \sqrt{\frac{g \sum w_i \cdot \delta_i}{\sum w_i \cdot \delta_i^2}}$$

$$\omega_n = \sqrt{\frac{5 \cdot g}{4 \cdot \delta_{st}}}$$

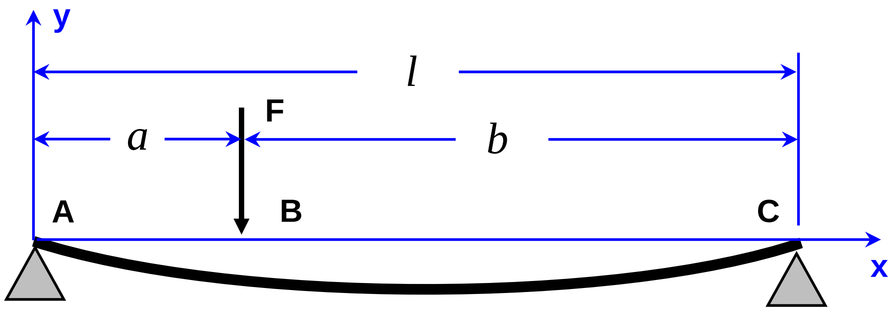
$$\delta_{st} = \frac{5 \cdot w \cdot L^4}{384 \cdot E \cdot I}$$

Rayleigh's Method

Beam Bending Tables

$$y_{AB} = \frac{F \cdot b \cdot x}{6 \cdot E \cdot I \cdot l}\left(x^2 + b^2 - l^2\right)$$

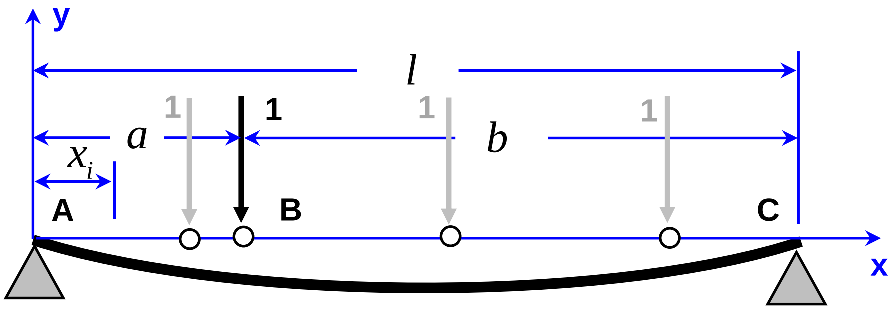
$$y_{BC} = \frac{F \cdot a \cdot (l - x)}{6 \cdot E \cdot I \cdot l}\left(x^2 + a^2 - 2 \cdot l \cdot x\right)$$



Influence Coefficients

$$a_{ij} = \left| \frac{1 \cdot b_i \cdot x_i}{6 \cdot E \cdot I \cdot l}\left(x^2 + b_i^2 - l^2\right) \right| \quad x_i \le a_i$$

$$a_{ij} = \left| \frac{1 \cdot a \cdot (l - x)}{6 \cdot E \cdot I \cdot l}\left(x^2 + a^2 - 2 \cdot l \cdot x\right) \right| \quad x_i > a_i$$



i^{th} Locaton on the Shaft due to the

j^{th} Load

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \\ & \vdots & & \ddots \end{bmatrix}$$

$$\omega_n \approx \sqrt{\frac{g\left(w_1 \cdot \delta_1 + w_2 \cdot \delta_2 + \cdots\right)}{w_1 \cdot \delta_1^2 + w_2 \cdot \delta_2^2 + \cdots}}$$

$$\delta_1 = w_1 \cdot a_{11} + w_2 \cdot a_{12} + w_3 \cdot a_{13} + \cdots$$
$$\delta_2 = w_1 \cdot a_{21} + w_2 \cdot a_{22} + w_3 \cdot a_{23} + \cdots$$
$$\delta_3 = w_1 \cdot a_{31} + w_2 \cdot a_{32} + w_3 \cdot a_{23} + \cdots$$
$$\vdots$$

Dunkerley's Method

Beam Bending Tables

$$y_{AB} = \frac{F \cdot b \cdot x}{6 \cdot E \cdot I \cdot l}\left(x^2 + b^2 - l\right)$$

$$y_{AB} = \frac{F \cdot a \cdot (l - x)}{6 \cdot E \cdot I \cdot l}\left(x^2 + a^2 - 2 \cdot l \cdot x\right)$$

Influence Coefficients

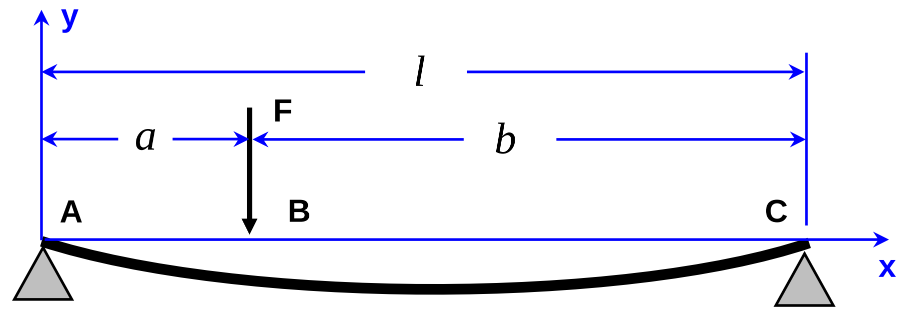
$$a_{ij} = \left| \frac{1 \cdot b_i \cdot x_i}{6 \cdot E \cdot I \cdot l}\left(x^2 + b_i^2 - l^2\right) \right| \quad x_i \le a_i$$

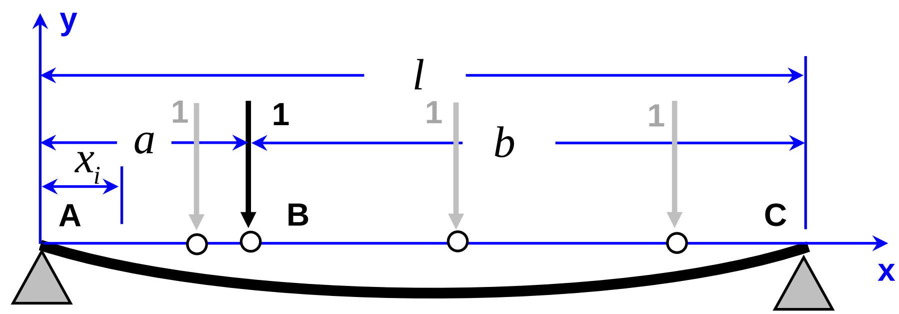
$$a_{ij} = \left| \frac{1 \cdot a \cdot (l - x)}{6 \cdot E \cdot I \cdot l}\left(x^2 + a^2 - 2 \cdot l \cdot x\right) \right| \quad x_i > a_i$$

i^{th} Locaton on the Shaft due to the

j^{th} Load

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \\ & \vdots & & \ddots \end{bmatrix}$$





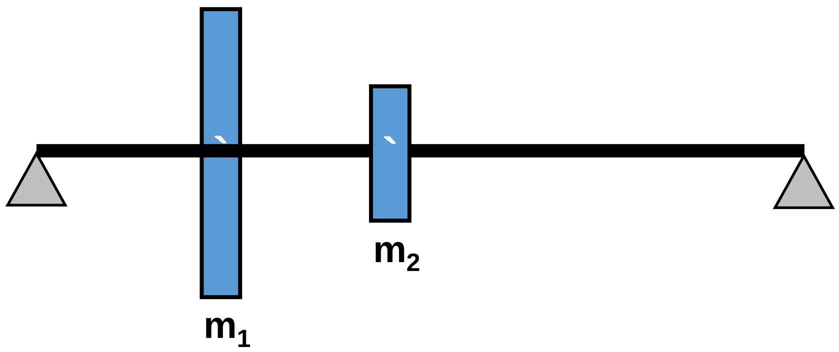
$$\frac{1}{\omega_c^2} \approx \sum \frac{1}{\omega_i^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots$$

$\omega_i \equiv$ The Natural Frequency if Load i is acting alone

$$= \sqrt{\frac{g}{\delta_{ii}}} = \sqrt{\frac{g}{w_i \cdot a_{ii}}}$$

Example

Two fixed masses on a solid shaft are simply supported at both ends. The masses, m_1 and m_2, weigh 70lb and 30lb respectively. Determine the critical speed of the shaft.



$$i^{th} \text{ Locaton on the Shaft due to the}$$

$$j^{th} \text{ Load}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3.4\times10^{-6}\ {}^{in}\!/\!_{lb} & 6.8\times10^{-6}\ {}^{in}\!/\!_{lb} \\ 6.8\times10^{-6}\ {}^{in}\!/\!_{lb} & 20.4\times10^{-6}\ {}^{in}\!/\!_{lb} \end{bmatrix}$$

Dunkerley Solution

The shaft deflections at each position due to each mass individually.

$$\delta_{11} = w_1 \cdot a_{11} = (70lb)\cdot\left(3.4\times10^{-6}\ {}^{in}\!/\!_{lb}\right) = 2.38\times10^{-4}\,in$$

$$\delta_{22} = w_2 \cdot a_{22} = (30lb)\cdot\left(20.4\times10^{-6}\ {}^{in}\!/\!_{lb}\right) = 6.12\times10^{-4}\,in$$

The shaft natural frequency due to each mass individually.

$$\omega_1 = \sqrt{\frac{g}{\delta_{11}}} = \sqrt{\frac{386\ {}^{in}\!/\!_{s^2}}{2.38\times10^{-6}\,in}} = 1273.5\,\frac{rad}{s}$$

$$\omega_2 = \sqrt{\frac{g}{\delta_{22}}} = \sqrt{\frac{386\ {}^{in}\!/\!_{s^2}}{6.12\times10^{-6}\,in}} = 794.2\,\frac{rad}{s}$$

Calculating the Critical Frequency using the DUNKERLEY equation (underestimate).

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \frac{1}{\left(1273.5\ {}^{rad}\!/\!_{s}\right)^2} + \frac{1}{\left(794.2\ {}^{rad}\!/\!_{s}\right)^2}$$

$$\boxed{\omega_c = 673.0\ {}^{rad}\!/\!_{s}}$$

Rayleigh Solution

Total shaft deflections at each.

$$\delta_1 = w_1 \cdot a_{11} + w_2 \cdot a_{12} = (70lb)\cdot\left(3.4\times10^{-6}\ {}^{in}\!/_{lb}\right) + (30lb)\cdot\left(6.8\times10^{-6}\ {}^{in}\!/_{lb}\right) = 4.42\times10^{-4}\,in$$

$$\delta_2 = w_1 \cdot a_{21} + w_2 \cdot a_{22} = (70lb)\cdot\left(6.8\times10^{-6}\ {}^{in}\!/_{lb}\right) + (30lb)\cdot\left(20.4\times10^{-6}\ {}^{in}\!/_{lb}\right) = 1.088\times10^{-3}\,in$$

Calculating the Critical Frequency using the RAYLEIGH equation (overestimate).

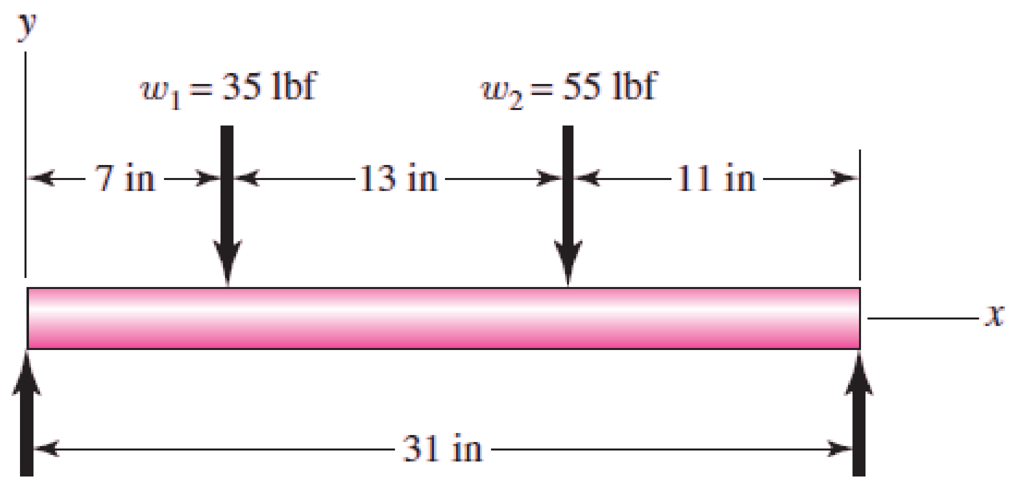
$$\omega_c = \sqrt{\frac{g\left(w_1\cdot\delta_1 + w_2\cdot\delta_2\right)}{w_1\cdot\delta_1^2 + w_2\cdot\delta_2^2}}$$

$$= \sqrt{\frac{386\ {}^{in}\!/_{s^2}\left[(70lb)\cdot\left(4.42\times10^{-4}\,in\right) + (30lb)\cdot\left(1.088\times10^{-3}\right)\right]}{(70lb)\cdot\left(4.42\times10^{-4}\,in\right)^2 + (30lb)\cdot\left(1.088\times10^{-3}\right)^2}}$$

$$\boxed{= 706.3\ {}^{rad}\!/_{s}}$$

Example

Consider a simple supported steel shaft with 1 in diameter and a 31 in span between bearings, carrying two gears weighing 35 and 55 lb. Estimate the critical frequency.



(a)

Dunkerley Solution

The shaft deflections at each position due to each mass individually.

$$\delta_{11} = w_1 \cdot a_{11} = \left(35lb\right)\cdot\left(206.1\times10^{-6}\ {}^{in}\!/\!_{lb}\right) = 7.212\times10^{-3}\,in$$

$$\delta_{22} = w_2 \cdot a_{22} = \left(55lb\right)\cdot\left(353.4\times10^{-6}\ {}^{in}\!/\!_{lb}\right) = 19.44\times10^{-3}\,in$$

The shaft natural frequency due to each mass individually.

$$\omega_1 = \sqrt{\frac{g}{\delta_{11}}} = \sqrt{\frac{386\ {}^{in}\!/\!_{s^2}}{7.212\times10^{-3}\,in}} = 231.3\,\frac{rad}{s}$$

$$\omega_2 = \sqrt{\frac{g}{\delta_{22}}} = \sqrt{\frac{386\ {}^{in}\!/\!_{s^2}}{19.44\times10^{-3}\,in}} = 140.9\,\frac{rad}{s}$$

Calculating the Critical Frequency using the DUNKERLEY equation (underestimate).

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \frac{1}{\left(231.3\ {}^{rad}\!/\!_{s}\right)^2} + \frac{1}{\left(140.9\ {}^{rad}\!/\!_{s}\right)^2}$$

$$\boxed{\omega_c = 120.3\ {}^{rad}\!/\!_{s}}$$

Rayleigh Solution

Total shaft deflections at each.

$$\delta_1 = w_1 \cdot a_{11} + w_2 \cdot a_{12} = (35lb) \cdot \left(206.1 \times 10^{-6}\ {}^{in}\!/\!_{lb}\right) + (55lb) \cdot \left(222.4 \times 10^{-6}\ {}^{in}\!/\!_{lb}\right) = 19.44 \times 10^{-3}\ in$$

$$\delta_2 = w_1 \cdot a_{21} + w_2 \cdot a_{22} = (35lb) \cdot \left(353.4 \times 10^{-6}\ {}^{in}\!/\!_{lb}\right) + (55lb) \cdot \left(222.4 \times 10^{-6}\ {}^{in}\!/\!_{lb}\right) = 27.22 \times 10^{-3}\ in$$

Calculating the Critical Frequency using the RAYLEIGH equation (overestimate).

$$\omega_c = \sqrt{\frac{g\left(w_1 \cdot \delta_1 + w_2 \cdot \delta_2\right)}{w_1 \cdot \delta_1^2 + w_2 \cdot \delta_2^2}}$$

$$= \sqrt{\frac{386\ {}^{in}\!/\!_{s^2}\left[(35lb) \cdot \left(19.44 \times 10^{-3}\ in\right) + (55lb) \cdot \left(27.22 \times 10^{-3}\right)\right]}{(35lb) \cdot \left(19.44 \times 10^{-4}\ in\right)^2 + (55lb) \cdot \left(27.22 \times 10^{-3}\right)^2}}$$

$$\boxed{= 124.8\ {}^{rad}\!/\!_{s}}$$