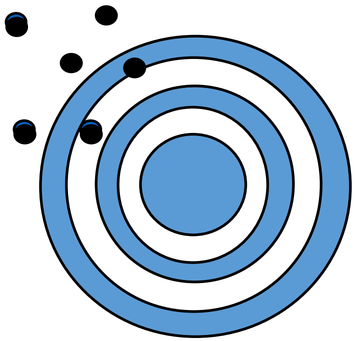
Uncertainty Analysis

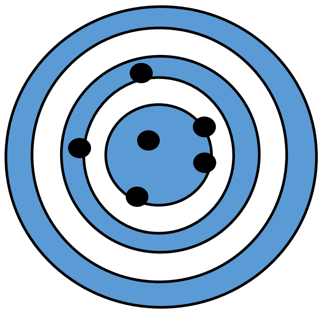
- **Types of Error**

- **Estimation of Uncertainty during**
  - **Design**
  - **Execution**
  - **Interpretation**

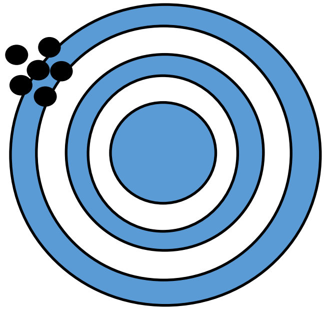
Relating Accuracy to a combination of Precision and Bias



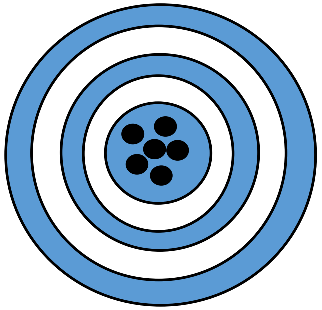
Biased, Not Precise



Not Biased, Not Precise



Biased, Precise



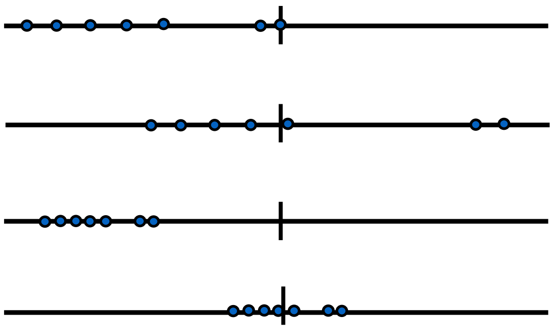
Not Biased, Precise = ACCURATE

Types of Error

- Measured Value Components

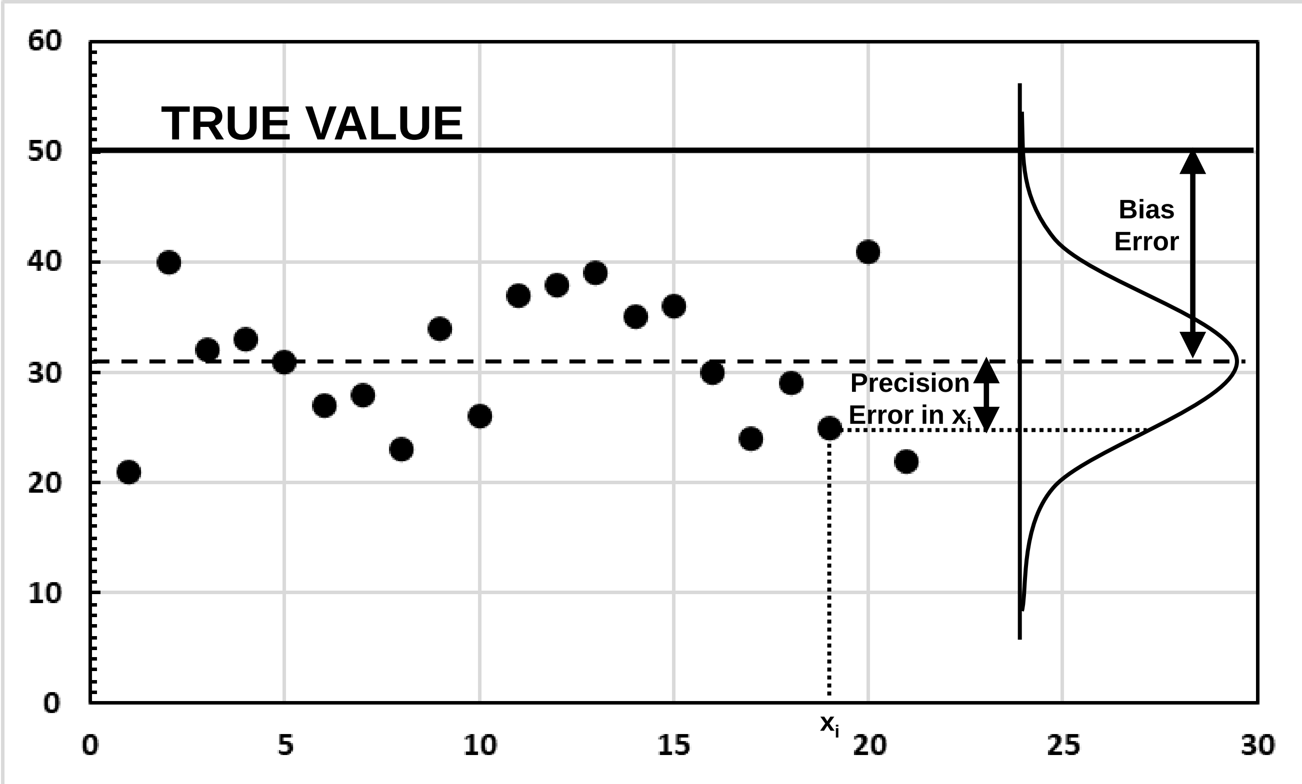
$$\text{Measured Value} = \text{True Value} + \text{Bias} + \text{Random Error}$$

$$\text{Measured Value} = \text{True Value} + \underset{(\mu - \text{True Value})}{\text{Bias}} + \underset{\sigma}{\text{Precision}}$$



- Bias: Fixed or Systematic
  - Assumed to be small
  - Equipment Calibrated
- Precision (Uncertainty): Random Error
  - Accumulated
- ACCURACY deals with the difference between the *true* value and the *measured* value
  - An ACCURATE measurement has both small BIAS and PRECISION errors.

Best Estimate of True Value, Sample Mean plus Uncertainty



$$x = \overline{x} \pm u_a$$

$$u_x \equiv bias + uncertainty\ in\ x$$

$$\overline{x} \equiv sample\ mean$$

$$x \equiv true\ value\ extimate\ without\ bias$$

Error Estimates

If an error is statistically estimated, treat it as a PRECISION error; otherwise, treat it as a BIAS error

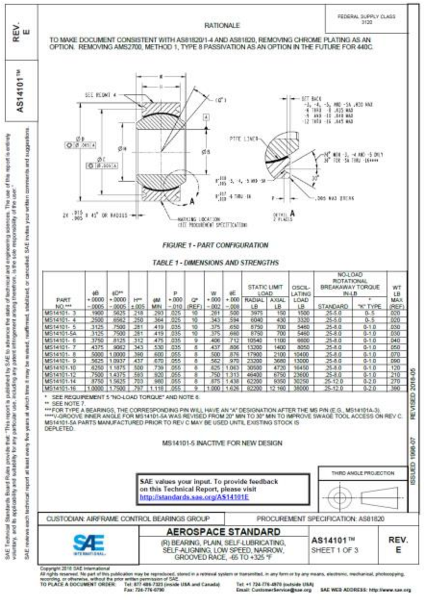
BIAS

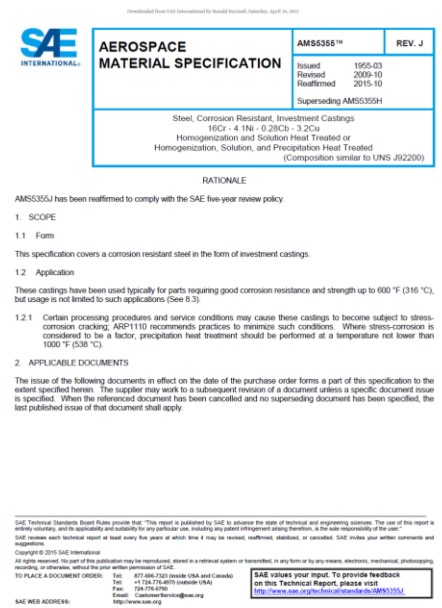
- **Can not be discerned by statistical means alone**
- **Each measurement contains the same amount of Bias**
- **Bias can only be estimated by comparison**
  - **Calibration**
  - **Concomitant Methodologies**
  - **Interlaboratory comparisons**
  - **Experience**

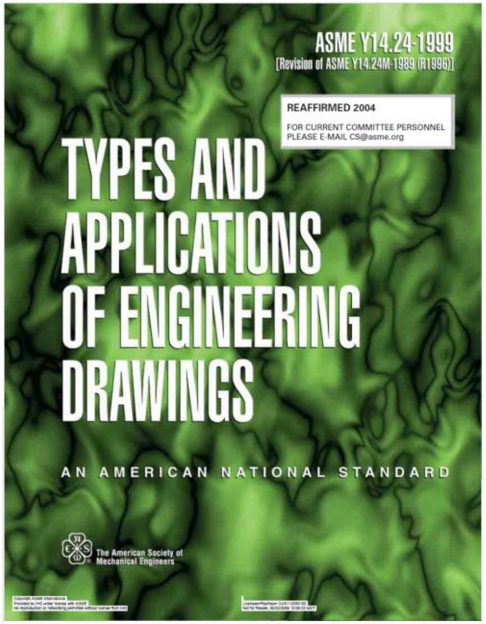
PRECISION

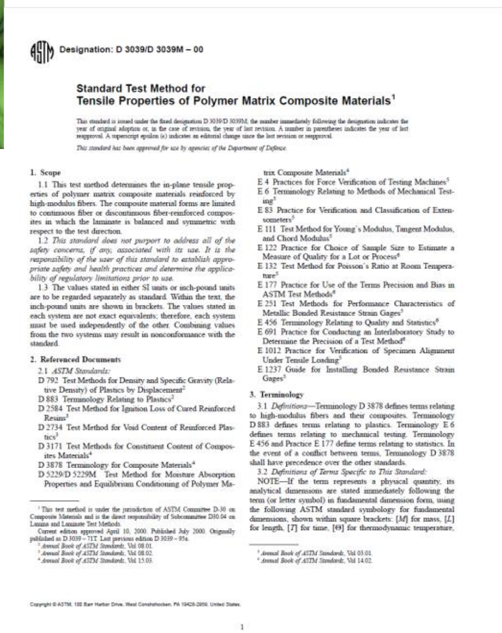
- **Can be statistically estimated**
- **Scatter in data generated under nominally fixed operating conditions**
  - **Measurement System**
    - **Repeatability and Resolution**
  - **Measured Variable**
    - **Temporal and Spatial Variations**
  - **Process**
    - **Variations in operating and environmental conditions**
  - **Measurement Procedure and Technique**
    - **Repeatability**

Standards


RATIONALE
FIGURE 1 - PART CONFIGURATION
TABLE 1 - DIMENSIONS AND STRENGTHS
AEROSPACE STANDARD
AS14101


AEROSPACE MATERIAL SPECIFICATION
AMS5355
REV. J
Steel, Corrosion Resistant, Investment Castings
RATIONALE
1. SCOPE
2. APPLICABLE DOCUMENTS


ASME Y14.24-1999
[Revision of ASME Y14.24M-1989 (R1996)]
REAFFIRMED 2004
FOR CURRENT COMMITTEE PERSONNEL
PLEASE E-MAIL CS@asme.org
TYPES AND APPLICATIONS OF ENGINEERING DRAWINGS
AN AMERICAN NATIONAL STANDARD
The American Society of Mechanical Engineers


Designation: D 3039/D 3039M – 00
Standard Test Method for
Tensile Properties of Polymer Matrix Composite Materials
1. Scope
2. Referenced Documents
3. Terminology

Uncertainty Analysis Assumptions

1. The test objectives are known.

2. The measurement is a clearly defined process in which all known CALIBRATION corrections for bias error have already been applied.

3. Data are obtained under fixed operating conditions

4. The engineers have some experience with the system components.
   a. Personal experience through previous tests or simulations
   b. Manufacturer’s literature
   c. NIST bulletins
   d. Professional Society codes and standards
      i. ASTM, ANSI, ASME, ASE, ISO, Etc.
   e. Technical literature

Stages of Uncertainty Analysis

1. **Design Stage Uncertainty**
   a. **An analysis of performance before the measurement**
   b. **Selecting instruments**
   c. **Selection measurement techniques**
   d. **Selection based on performance and cost**

2. **Advanced Stage and/or Single Measurement Uncertainty**

3. **Multiple Measurement Uncertainty analysis**

Contributions to Design Stage Uncertainty

- **Zero-Order Uncertainty, u_0**
  - **Uncertainty caused by the data scatter that results when the instrument is measured**
  - **Arbitrary Rule: one-half of the instrument's resolution with a probability of 95%**

$$u_0 = \pm \frac{1}{2} \cdot Resolution \ (95\%)$$

  - **1 TRUE value in 20 falls outside the u_0 range**

- **Manufacturer's statement concerning error, u_c**
  - **Catalog value for the type of instrument under ideal conditions.**

- **Errors combined using root-sum-squares (RSS) method**

$$u_d = \pm\sqrt{e_1^2 + e_2^2 + \cdots + e_k^2} \quad (95\%)$$

$$= \pm\sqrt{u_0^2 + u_c^2} \quad (95\%)$$

Example 1: Force Measurement

Consider the force-measuring instrument described by the catalog data that follows. Provide an estimate of the uncertainty attributable to this instrument and the instrument design stage uncertainty.

Resolution:	0.25 N
Range:	0-100 N
Linearity:	within 0.20 N over the range
Repeatability:	within 0.30 over the range

Solution 1: Force Measurement

Resolution:	0.25 N
Range:	0-100 N
Linearity:	within 0.20 N over the range
Repeatability:	within 0.30 over the range

The interpolation error (1/2 Resolution) u_0 is

$$u_o = {}^{0.25N}/_{2} = 0.125N \ \ (95\%)$$

The catalog uncertainty is due to both Linearity and Repeatability

$$u_c = \pm\sqrt{(0.2N)^2 + (0.3N)^2}$$
$$= \pm 0.36N \ \ (95\%)$$

The Design Stage Uncertainty is

$$u_d = \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{(0.125N)^2 + (0.36N)^2}$$
$$= \pm 0.38N \ \ (95\%)$$

Example 2: Voltmeter/Pressure Transducer

A voltmeter is to be used to measure the output from the pressure transducer that outputs an electrical signal. The normal pressure expected will be **~3 psi**. Estimate the design-stage uncertainty in this combination. The following information is available.

Voltmeter:

- **Resolution:** 10 μV
- **Accuracy:** within 0.001% of reading

Transducer

- Range: $\pm$ 5 psi
- **Sensitivity:** 1 V/psi
- Input Power: 10 $V_{dc} \pm 1\%$
- Output: $\pm$ 5 V
- **Linearity:** within 2.5 mV/psi over the range
- **Repeatability:** within 2 mV/psi over the range
- **Resolution:** Negligible

Solution 2: Voltmeter/Pressure Voltmeter

A voltmeter is to be used to measure the output from the pressure transducer that outputs an electrical signal. The normal pressure expected will be **~3 psi**. Estimate the design-stage uncertainty in this combination. The following information is available.

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Accuracy:	within 0.001% of reading

Transducer:

Resolution:	Negligible
Range:	± 5 psi
Sensitivity:	1 V/psi
Input Power:	10 V_{dc} ± 1%
Output:	± 5 V
Linearity:	within 2.5 mV/psi over the range
Repeatability:	within 2 mV/psi over the range

Uncertainty of the **VOLTMETER:** $u_d = \pm\sqrt{u_0^2 + u_c^2}$

Zeroth order Uncertainty: $u_0 = {}^{10\mu V}/_2 = \boxed{\pm 5\mu V \quad (95\%)}$

For a nominal pressure of ~3 psi and a sensitivity of 1V/psi, expected output 3V

$$u_c = 3V \cdot \underbrace{\frac{0.001\%}{100\%}}_{} = \boxed{\pm 30\mu V \quad (95\%)}$$

Translating a %
Into a decimal value

VOLTMETER Design Stage Uncertainty: $u_{dV} = \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{(\pm 5\mu V)^2 + (\pm 30\mu V)^2}$

$$= \boxed{\pm 30.4\mu V = \pm 0.0304\, mv \quad (95\%)} \quad ①$$

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- Resolution: Negligible
- Range: $\pm$ 5 psi
- **Sensitivity:** 1 V/psi
- Input Power: 10 V_{dc} $\pm$ 1%
- Output: $\pm$ 5 V
- **Linearity:** within 2.5 mV/psi over the range
- **Repeatability:** within 2 mV/psi over the range

Uncertainty of the PRESSURE TRANSDUCER

$$u_d = \pm\sqrt{u_0^2 + u_c^2}$$

$$u_0 = 0\ mv \quad (95\%)$$

For a nominal pressure of ~3 psi, using the linearity and repeatability, the **PRESSURE TRANSDUCER** design level uncertainty is

$$u_{dT} = \pm\sqrt{u_0^2 + \cancel{u_c^2}} = \sqrt{\left(2.5\ {}^{mv}/_{psi} \cdot 3psi\right)^2 + \left(2\ {}^{mv}/_{psi} \cdot 3psi\right)^2} = \boxed{\pm 9.61\ mv \quad (95\%)}$$ ②

The **TOTAL Design Stage Uncertainty: ① and ② combined**

$$u_d = \pm\sqrt{u_{dV}^2 + u_{dT}^2} = \pm\sqrt{(\pm 0.0304 mv)_{dV}^2 + (\pm 9.61\ mv)_{dT}^2} = \boxed{\pm 9.61\ mv \quad (95\%) = \pm 0.0096\ psi\ (95\%)}$$

Example 3: Tachometer

A tachometer has an analog display dial graduated in 5 revolutions per minute (rpm) increments. The user manual states an accuracy of 1% of reading. Estimate the uncertainty in the reading at 10, 500, and 5000 rpm.

Solution 3: Tachometer

A tachometer has an analog display dial graduated in 5 revolutions per minute (rpm) increments. The user manual states an accuracy of 1% of reading. Estimate the uncertainty in the reading at 10, 500, and 5000 rpm.

Resolution:	5 rpm
Accuracy:	within ±1% of reading

Zeroth Order Uncertainty u_0:

$$\boldsymbol{u_0 = \pm \frac{5\,rpm}{2} = \pm 2.5\,rpm \quad (95\%)}$$

Catalog Uncertainty u_c:

The accuracy is stated to be within ±1% of the reading. For readings at 10, 500, and 5000 rpm.

$$\boldsymbol{(u_c)_{10} = \pm 1\% \cdot 10\,rpm = 0.01 \cdot 10\,rpm = \pm 0.1\,rpm}$$
$$\boldsymbol{(u_c)_{500} = \pm 1\% \cdot 500\,rpm = 0.01 \cdot 500\,rpm = \pm 5.0\,rpm}$$
$$\boldsymbol{(u_c)_{5000} = \pm 1\% \cdot 5000\,rpm = 0.01 \cdot 5000\,rpm = \pm 50.0\,rpm}$$

Design Stage Uncertainty u_d:

Combining the zeroth order and catalog uncertainties using RSS

$$\boldsymbol{(u_d)_{10} = \pm\sqrt{(u_0)^2 + (u_c)_{10}^2} = \sqrt{(\pm 2.5\,rpm)^2 + (= \pm 0.1\,rpm)^2} = \boxed{\pm 2.5\,rpm \ \ (95\%)}}$$

$$\boldsymbol{(u_d)_{500} = \pm\sqrt{(u_0)^2 + (u_c)_{500}^2} = \sqrt{(\pm 2.5\,rpm)^2 + (= \pm 5.0\,rpm)^2} = \boxed{\pm 5.6\,rpm \ \ (95\%)}}$$

$$\boldsymbol{(u_d)_{5000} = \pm\sqrt{(u_0)^2 + (u_c)_{5000}^2} = \sqrt{(\pm 2.5\,rpm)^2 + (= \pm 50.0\,rpm)^2} = \boxed{\pm 50.0 rpm \ \ (95\%)}}$$

Example 4: Automobile

An automobile speedometer is graduated in 5-mph (8kph) increments and has an accuracy rated to be within ±4%. Estimate the uncertainty in indicated speed at 60 mph (90 kph)

Solution 4: Automobile

An automobile speedometer is graduated in 5-mph (8kph) increments and has an accuracy rated to be within ±4%. Estimate the uncertainty in indicated speed at 60 mph (90 kph)

Resolution: 5-mph (8kph)
Accuracy: within ±4%

Zeroth Order Uncertainty u_0:

$$\boldsymbol{u_0 = \pm \frac{5\,mph}{2} = \pm 2.5\,mph \quad (95\%) \; = \; 4\,kph \quad (95\%)}$$

Catalog Uncertainty u_c:

The accuracy is stated to be within ±4% of the reading. The indicated speed here is 60 mph (90 kph).

$$\boldsymbol{u_c = \pm 4\% \cdot 60\,mph = 0.04 \cdot 60\,mph = \pm 2.4\,mph\ @\ 60\,mph = \pm 3.6\,kph\ @\ 90\,kph}$$

Design Stage Uncertainty u_d:

Combining the two solutions using RSS

$$\boldsymbol{u_d = \pm\sqrt{(u_0)^2 + (u_c)^2} = \sqrt{(\pm 2.5\,mph)^2 + (= \pm 2.4\,mph)^2} = \boxed{\begin{array}{l} \pm 3.5\,mph @\ 60\,mph\ (95\%) \\ \pm 5.4 kph\ @\ 90\,kph\ (95\%) \end{array}}}$$

Example 5: Temperature

A temperature measurement system is composed of a sensor and a readout device. The readout device has an accuracy of 0.6°C with a resolution of 0.1°C. The sensor has an off-the-self accuracy of 0.5°C. Estimate the design-stage uncertainty in the temperature indicated by this combination of sensor and readout.

Solution 5: Temperature

A temperature measurement system is composed of a sensor and a readout device. The readout device has an accuracy of 0.6°C with a resolution of 0.1°C. The sensor has an off-the-self accuracy of 0.5°C. Estimate the design-stage uncertainty in the temperature indicated by this combination of sensor and readout.

Temperature Sensor
- Error Limit: ±0.5°C

Readout Device
- Resolution: 0.1°C
- Accuracy: 0.6°C

Zeroth Order Uncertainty u_0:

Sensor

$$\boldsymbol{u_{0s} = \pm 0 \quad (no\ readout)}$$

Readout

$$\boldsymbol{u_0 = \pm \frac{0.1°C}{2} = \pm 0.05\text{℃} \ (95\%)}$$

Catalog Uncertainty u_c:

Sensor

$$\boldsymbol{u_{cs} = \pm 0.5\ \text{℃}}$$

Readout

$$\boldsymbol{u_{cR} = \pm 0.6\ \text{℃}}$$

Design Stage Uncertainty u_d:

Combining all the uncertainties using RSS

$$\boldsymbol{u_d = \pm\sqrt{(u_{0s})^2 + (u_{cs})^2 + (u_{0R})^2 + (u_{cR})^2}}$$

$$\boldsymbol{= \pm\sqrt{(\pm 0\text{℃})^2 + (\pm 0.05\text{℃})^2 + (\pm 0.05\text{℃})^2 + (\pm 0.6\text{℃})^2} = \boxed{\pm 0.78\text{℃} \ (95\%)}}$$

Example 6: Pressure Readout

An equipment catalog boasts that a pressure transducer system comes in 3 ½ digit (e.g., 19.99) or 4 ½ digit (e.g., 19.999) display (the half digit is the leading 1, this digit can only display the 1). The 4 ½ digit model costs 50% more. Both units are otherwise identical. The specifications are as follows:

Linearity error: 0.15% FSO
Hysteresis error: 0.20% FSO
Repeatability error: 0.25% FSO

(FS0 $\equiv$ Full Scale Output). For an FSO of 20 kPa, select a readout based on appropriate uncertainty calculations.

Solution 6: Pressure Readout

An equipment catalog boasts that a pressure transducer system comes in 3 ½ digit (e.g., 19.99) or 4 ½ digit (e.g., 19.999) display. The 4 ½ digit model costs 50% more. Both units are otherwise identical. The specifications are as follows:

Linearity error: 0.15% FSO → $u_1 = 0.0015 \cdot 20$ kPa = 0.03 kPa
Hysteresis error: 0.30% FSO → $u_2 = 0.002 \cdot 20$ kPa = 0.04 kPa
Repeatability error: 0.25% FSO → $u_3 = 0.0025 \cdot 20$ kPa = 0.05 kPa

(FS0 ≡ Full Scale Output). For an FSO of 20 kPa, select a readout based on appropriate uncertainty calculations.

Zeroth Order Uncertainty u_0:

3 ½ Readout

$$u_{0\,3^1/_2} = \mathbf{0.01}\ kPa \quad (95\%)$$

4 ½ Readout

$$u_{0\,4^1/_2} = \mathbf{0.001}\ kPa \quad (95\%)$$

Catalog Uncertainty u_c: (Same for both readout types)

Linearity error: 0.15% FSO → $u_1 = 0.0015 \cdot 20$ kPa = 0.03 kPa
Hysteresis error: 0.30% FSO → $u_2 = 0.002 \cdot 20$ kPa = 0.04 kPa
Repeatability error: 0.25% FSO → $u_3 = 0.0025 \cdot 20$ kPa = 0.05 kPa

$$u_c = \pm\sqrt{(u_{c1})^2 + (u_{c2})^2 + (u_{c3})^2} = \pm\sqrt{(\mathbf{0.03}\ kPa)^2 + (\mathbf{0.04}\ kPa)^2 + (\mathbf{0.05}\ kPa)^2} = \pm\mathbf{0.0707}\ kPa \ \ (95\%)$$

Design Stage Uncertainty u_d:

Combining all the uncertainties using RSS

3 ½ Readout

$$u_{d\,3^1/_2} = \pm\sqrt{\left(u_{0\,3^1/_2}\right)^2 + (u_c)^2}$$
$$= \pm\sqrt{(\mathbf{0.01}\ kPa)^2 + (\mathbf{0.0707}\ kPa)^2}$$
$$= \pm\mathbf{0.0714}\ kPa \ \ (95\%)$$

4 ½ Readout

$$u_{d\,4^1/_2} = \pm\sqrt{\left(u_{0\,4^1/_2}\right)^2 + (u_c)^2}$$
$$= \pm\sqrt{(\mathbf{0.001}\ kPa)^2 + (\mathbf{0.0707}\ kPa)^2}$$
$$= \boxed{\pm\mathbf{0.0707}\ kPa \ \ (95\%)}$$

The two readouts have the SAME Design Level Uncertainty for all practical purposes.

Example 7: Crank Shaft

You are to measure the rotational speed of a crankshaft. The available angular velocity meter has a display with a resolution of 2 rpm (revolutions/minute). The meter specifications indicate the device's accuracy is 1.5% of the velocity reading. Find the design stage uncertainty for each velocity: a) 40 rpm, b)400 rpm, and c)4000 rpm.

Example 8: A/D Converter

A temperature measurement system is comprised of a low-voltage temperature sensor, an anti-aliasing filter, and an A/D converter. The sensor's output voltage is linearly proportional to the temperature with a sensitivity of 10mV/°C. The operational amplifier gains the sensor voltage by a factor of 10.

The manufacturers of the instruments provided the following accuracies and information.

Temperature Sensor:	1%
Operational Amplifier:	2%
Anti-Aliasing Filter:	0.5%
A/D Converter:	1%: 12 bit; 10 V full scale resolution

Determine the zero-th order uncertainty and the design stage uncertainty.

Solution 8: A/D Converter

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Zeroth Order Uncertainty:

Resolution of a 12 bit A/D converter:

- 12 bit Resolution = 1 part of 2^{12} = 1 part of 4096

$$Resolution = \frac{Full\ Scale\ Range}{2^{12}\ bits} = \frac{10V}{4096\ bits} = 2.44\,\frac{mV}{bit}$$

$$u_0 = \frac{bit}{2} = \underbrace{\frac{bit \cdot 2.44\,{}^{mV}/_{bit}}{2}}_{\text{Conversion bit to mV}} = 1.22\ mV \cdot \underbrace{\left(\frac{0.001V}{mv}\right)}_{\text{Conversion mV to V}} = 0.00122\ \text{V} \quad ①$$

- As a percentage of Full Scale

$$u_{0\ \%FS} = Percent\ Full\ Scale\ Zeroth\ Order\ Uncertainty = 0.00122\ V \cdot \left(\frac{100\%}{10\ V}\right) = \boxed{0.0122\%} \quad ②$$

Solution 8: A/D Converter

A temperature measurement system is comprised of a low-voltage temperature sensor, an anti-aliasing filter, and an A/D converter. The sensor's output voltage is linearly proportional to the temperature with a sensitivity of 10mV/°C. The operational amplifier gains the sensor voltage by a factor of 10.

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Temperature Sensor:	1%
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Design Stage Uncertainty:

RSS Calculation in % Uncertainty:

$$\boldsymbol{u_{d\%} \equiv \pm Design\ Stage\ Uncertainty = \pm\sqrt{u_0^2 + u_{sensor}^2 + u_{amplifier}^2 + u_{filter}^2 + u_{converter}^2}}$$

$$= \pm\sqrt{(\mathbf{0.0122}\%)^2 + (\mathbf{1.0}\%)^2 + (\mathbf{2.0}\%)^2 + (\mathbf{0.5}\%)^2 + (\mathbf{1.0}\%)^2} = \boxed{\pm\mathbf{2.5}\%}$$

$$\boldsymbol{u_d = Design\ Stage\ Uncertainty = \pm u_{d\%} \cdot \left(\frac{10\ V}{100\%}\right) = \pm 2.5\% \cdot \left(\frac{10\ V}{100\%}\right) = \boxed{\pm 0.250\ V}}$$

Stages of Uncertainty Analysis

1. Design Stage Uncertainty

2. Advanced Stage and/or Single Measurement Uncertainty

3. Multiple Measurement Uncertainty analysis

Estimation of Uncertainty

- The goal is to estimate the ***uncertainty*** of experimental measurements and calculate results due to random error
- Procedure
  - Estimate the ***uncertainty interval*** $\pm 2\sigma$ for each measured quantity
  - State the ***confidence limit*** on each measurement
    - Random Measurement error is assumed Normally Distributed
  - Analyze the propagation of uncertainty into results calculated from experimental data

Estimate the Measurement Uncertainty Interval

- Measured Variables: x_1, x_2, …

- Estimate Random Error
  - $\pm n\sigma$ where **n=1,2,3…**
  - $\pm 3\sigma$: 99 percent of measured values
  - $\pm 2\sigma$: 95 percent of measured values
  - $\pm \sigma$: 68 percent of measured values

- Single Sample
  - ± half the smallest scale division (the least count) of the instrument

Analyze the Propagation of Uncertainty in Calculations

- Measurements of independent variables in lab, x_1, x_2, …
- The relative uncertainty of each independently measured quantity is estimated as u_i
- The measurements are used to calculate the result, R
- We wish to estimate how the errors in the x_i's propagate into the calculation of R

Calculation of Uncertainty - DIRECT
Kline-McClintock Second Power Relation

Every Regular Equation has an error equation.

Every Error Equation has one term for each measured quantity

$$\boldsymbol{R = R(x_1, x_2, \cdots, x_n)}$$

$$\boldsymbol{\delta R_i = \frac{\partial R}{dx_i} \cdot \delta x_i} \qquad \boldsymbol{variation}$$

$$\boldsymbol{u_{R_i} = \frac{\partial R}{dx_i} \cdot u_{x_i}}$$

$$\boldsymbol{u_{R_i} \equiv x_i's\ \ contribution\ to\ the\ uncertainty\ of\ R}$$

$$\boldsymbol{u_{x_i} \equiv x_i's\ \ uncertainty}$$

Kline, SJ and McClintock, FA, “Describing uncertainties in single sample experiments,” Mechanical Engineering (75), 1953.

Uncertainty Interval - DIRECT

- Single Variable

$$\boldsymbol{u_{R_i} = \frac{\partial R}{\partial x_i} \cdot u_{x_i}}$$

- Multiple Variables

$$\boldsymbol{R = R(x_1, x_2, \cdots, x_n)}$$

$$\boldsymbol{\delta R_i = \frac{\partial R}{\partial x_i} \cdot \delta x_i} \qquad \boldsymbol{variation}$$

$$\boldsymbol{u_{R_i} = \frac{\partial R}{\partial x_i} \cdot u_{x_i}}$$

  - Absolute Uncertainty

$$\boldsymbol{u_R = \pm\left[\left(\frac{\partial R}{\partial x_1} \cdot u_{x_1}\right) + \left(\frac{\partial R}{\partial x_2} \cdot u_{x_2}\right) + \cdots + \left(\frac{\partial R}{\partial x_i} \cdot u_{x_i}\right) + \cdots + \left(\frac{\partial R}{\partial x_n} \cdot u_{x_n}\right)\right]}$$

  - Constant Odds Uncertainty – Accepted Method

$$\boldsymbol{u_R = \pm\left[\left(\frac{\partial R}{\partial x_1} \cdot u_{x_1}\right)^2 + \left(\frac{\partial R}{\partial x_2} \cdot u_{x_2}\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_i} \cdot u_{x_i}\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_n} \cdot u_{x_n}\right)^2\right]^{1/2}}$$

Uncertainty Interval - NORMALIZED

$$R = R(x_1, x_2, \cdots, x_n)$$

$$\delta R_i = \frac{\partial R}{\partial x_i} \cdot \delta x_i \xrightarrow[\text{sides by } 1/R]{\text{Multiply both}} \frac{\delta R_i}{R} = \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{R}$$

$$\xrightarrow[\text{by 1 in the form } x_i/x_i]{\text{Multiply the Right-Hand side}} \frac{\delta R_i}{R} = \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{R} \cdot \frac{x_i}{x_i} = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{x_i}$$

$$\frac{\delta R_i}{R} \equiv \textit{The normalized contribution to the uncertainty of R by } x_i = U_i$$

$$\frac{\delta x_i}{x_i} \equiv \textit{The normalized uncertainty of } x_i = U_{x_i}$$

$$\frac{\delta R_i}{R} = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{x_i} \longrightarrow U_i = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot U_{x_i}$$

$$U_R \equiv \textit{the normalized uncertainty of } R = \frac{\delta R}{R} = \pm \left[\sum_{i=1}^{n} U_i^2\right]^{1/2}$$

$$= \pm \left[\left(\frac{x_1}{R} \cdot \frac{\partial R}{\partial x_1} \cdot U_{x_1}\right)^2 + \left(\frac{x_2}{R} \cdot \frac{\partial R}{\partial x_2} \cdot U_{x_2}\right)^2 + \cdots + \left(\frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot U_{x_i}\right)^2 + \cdots + \left(\frac{x_n}{R} \cdot \frac{\partial R}{\partial x_n} \cdot U_{x_n}\right)^2\right]^{1/2}$$

Kline, SJ and McClintock, FA, “Describing uncertainties in single sample experiments,” Mechanical Engineering (75), 1953.

Example 9 : Volume Uncertainty, Analytical Approach

- Determine the Uncertainty in the Volume of Cylinder
  - $d = 2.9 \pm 0.16$ cm
  - $h = 1.5 \pm 0.11$ cm
  - $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

First, consider the extremes in the measurement of volume.

Next, consider both analytical approaches (Direct and Normalized)

Solution 9 : DIRECT Approach

- Determine the Uncertainty in the Volume of Cylinder
  - d = 2.9 ± 0.16 cm
  - h = 1.5 ± 0.11 cm
  - V = V(d,h) = ¼·**π**·d²·h

Extreme Value

	−Δ	+Δ
d	2.74 cm	3.06 cm
h	1.39 cm	1.61 cm

Analytical Approach – Direct

$$V = \frac{\pi}{4} \cdot d^2 \cdot h = \frac{\pi}{4} \cdot (2.9\ cm)^2 \cdot (1.5\ cm) = 9.91\ cm$$

$$u_V = \pm\left[\left(\frac{\partial V}{\partial d} \cdot u_d\right)^2 + \left(\frac{\partial V}{\partial h} \cdot u_h\right)^2\right]^{1/2} = \pm\left[\left(\frac{\partial\left(\frac{\pi}{4} \cdot d^2 \cdot h\right)}{\partial d} \cdot u_d\right)^2 + \left(\frac{\partial\left(\frac{\pi}{4} \cdot d^2 \cdot h\right)}{\partial h} \cdot u_h\right)^2\right]^{1/2}$$

$$= \pm\left[\left(\frac{\pi}{4} 2 \cdot d \cdot h \cdot u_d\right)^2 + \left(\frac{\pi}{4} \cdot d^2 \cdot u_h\right)^2\right]^{1/2}$$

$$= \pm\left[\left(\frac{\pi}{2} \cdot 2.9\ cm \cdot 1.5\ cm \cdot 0.16\ cm\right)^2 + \left(\frac{\pi}{4} \cdot (2.9\ cm)^2 \cdot 0.11\ cm\right)^2\right]^{1/2}$$

$$= \boxed{\pm 1.31\ cm^3\ \ (95\%) \implies V = 9.91 \pm 1.31\ cm^3 = 8.60\ cm^3 \rightarrow 11.22\ cm^3}$$

Solution 9: Volume Uncertainty-Normalized

- **Determine the Uncertainty in the Volume of Cylinder**
  - **d = 2.9 ± 0.16 cm**
  - **h = 1.5 ± 0.11 cm**
- **V = V(d,h) = ¼·π·d²·h**

$$V = \frac{\pi}{4} \cdot d^2 \cdot h = \frac{\pi}{4} \cdot (2.9\ cm)^2 \cdot (1.5\ cm) = 9.91\ cm^3$$

$$\frac{\delta d}{d} = \frac{\pm 0.16\ cm}{2.9\ cm} = \pm 0.0552 = U_d$$

$$\frac{\delta h}{h} = \frac{\pm 0.11\ cm}{1.5\ cm} = \pm 0.0733 = U_d$$

$$\frac{u_V}{V} = U_V = \pm\left[\left(\frac{d}{V} \cdot \frac{\partial V}{\partial d} \cdot U_d\right)^2 + \left(\frac{h}{V} \cdot \frac{\partial V}{\partial h} \cdot U_h\right)^2\right]^{1/2}$$

$$= \pm\left[\left(\frac{d}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\partial\left(\frac{\pi}{4} \cdot d^2 \cdot h\right)}{\partial d} \cdot U_d\right)^2 + \left(\frac{h}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\partial\left(\frac{\pi}{4} \cdot d^2 \cdot h\right)}{\partial h} \cdot U_h\right)^2\right]^{1/2}$$

$$= \pm\left[\left(\frac{d}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\pi}{4} 2 \cdot d \cdot h \cdot U_d\right)^2 + \left(\frac{h}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\pi}{4} \cdot d^2 \cdot U_h\right)^2\right]^{1/2} = \pm\left[(2 \cdot U_d)^2 + (U_h)^2\right]^{1/2}$$

Exactly the same as the previous result

$$= \pm\left[(0.1104)^2 + (0.0733)^2\right]^{1/2} = \pm 0.133 \quad \Longrightarrow \quad u_V = V \cdot U_V = \left(9.91\ cm^3\right) \cdot (\pm 0.133) = \boxed{1.31 cm^3\ (95\%)}$$

If the Equation is Too Complex

Sequential Numerical Perturbation

1. **Based on measurements for the independent variables under some fixed operating condition, calculate a result R_0 where $R_0=f(x_1, x_2,\ldots, x_N)$. This value fixes the operating point for the numerical approximation.**

2. **Next, increase the independent variables by their respective uncertainties and recalculate the results based on each of these new values. Call these values R_i^+. That is**
   - $R_1^+=f(x_1+u_{x1}, x_2,\ldots, x_N)$
   - $R_2^+=f(x_1, x_2+u_{x2},\ldots, x_N)$
   - $R_N^+=f(x_1, x_2,\ldots, x_N+u_{xN})$

3. **Next, similarly, decrease the independent variables by their respective uncertainties and recalculate the results based on each of these new values. Call these values R_i^-.**

4. **Finally, evaluate the approximation of the uncertainty contributions from each variable.**

$$\delta R_i = \frac{|\delta R_i^+| - |\delta R_i^-|}{2} \approx \theta_i \cdot u_i$$

$$u_R = \pm\left[\sum_{i=1}^{n}(\delta R_i)^2\right]^{1/2} \qquad (P\%)$$

Example 10: Volume Uncertainty, Numerical Approach

- Determine the Uncertainty in the Volume of Cylinder
  - $d = 29 \pm 0.16$ cm
  - $h = 1.5 \pm 0.11$ cm

- $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

Solution 10: Volume Uncertainty, Numerical Approach

- Determine the Uncertainty in the Volume of Cylinder
  - d = 2.9 ± 0.16 cm
  - h = 1.5 ± 0.11 cm
- V = V(d,h) = ¼·π·d²·h

$$V = \frac{\pi}{4} \cdot d^2 \cdot h$$

$$V_d^+ = \frac{\pi}{4} \cdot (2.9\ cm + 0.16\ cm)^2 \cdot (1.5\ cm) = 11.03\ cm^3$$

$$V_d^- = \frac{\pi}{4} \cdot (2.9\ cm - 0.16\ cm)^2 \cdot (1.5\ cm) = 8.84\ cm^3$$

$$\delta V_d = \frac{|V_d^+| - |V_d^-|}{2} = \frac{|11.03\ cm^3| - |8.84\ cm^3|}{2} = 1.094\ \ cm^3$$

$$V_h^+ = \frac{\pi}{4} \cdot (2.9\ cm)^2 \cdot (1.5\ cm + 0.11\ cm) = 10.63\ cm^3$$

$$V_h^- = \frac{\pi}{4} \cdot (2.9\ cm)^2 \cdot (1.5\ cm - 0.11\ cm) = 9.18\ cm^3$$

$$\delta V_h = \frac{|V_h^+| - |V_h^-|}{2} = \frac{|10.63\ cm^3| - |9.18\ cm^3|}{2} = 0.73\ cm^3$$

Exactly the same as the previous results

$$u_V = \pm\left[\left(1.094\ cm^3\right)^2 + \left(0.73\ cm^3\right)^2\right]^{1/2} = \boxed{\pm 1.31\ cm^3 (95\%)}$$

The analytical and numerical approaches yield the same result. This will not always be the case.

Example 11: Liquid Mass Flow Rate

The mass flow rate of water through a tube is determined by collecting water in a beaker. The mass flow rate is calculated from the net mass in the water collected divided by the time interval. Determine the uncertainty of the mass flow rate.

- Mass Flow Rate

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

$$\Delta m = m_f - m_e$$

- Error Estimates for measured Quantitates
  - M_f=400±2g (20 to 1)
  - M_e=200±2g (20 to 1)
  - Δt=10±0.2s (20 to 1)

Solution 11: Liquid Mass Flow Rate

- Mass Flow Rate
- Error Estimates for measured Quantitates
  - M_f=400±2g (20 to 1)
  - M_e=200±2g (20 to 1)
  - Δt=10±0.2s (20 to 1)

$$\dot{m} = \frac{\Delta m}{\Delta t}$$
$$\Delta m = m_f - m_e$$

$$\dot{m} = \frac{\Delta m}{\Delta t} = \frac{m_f - m_e}{\Delta t} = \frac{(400g) - (200g)}{(10s)} = 20\,\frac{g}{s}$$

$$u = \pm\sqrt{\left(\frac{\partial \dot{m}}{\partial m_f}\cdot u_{m_f}\right)^2 + \left(\frac{\partial \dot{m}}{\partial m_e}\cdot u_{m_e}\right)^2 + \left(\frac{\partial \dot{m}}{\partial \Delta t}\cdot u_{\Delta t}\right)^2} = \sqrt{\left(\frac{\partial\left(\frac{m_f - m_e}{\Delta t}\right)}{\partial m_f}\cdot u_{m_f}\right)^2 + \left(\frac{\partial\left(\frac{m_f - m_e}{\Delta t}\right)}{\partial m_e}\cdot u_{m_e}\right)^2 + \left(\frac{\partial\left(\frac{m_f - m_e}{\Delta t}\right)}{\partial \Delta t}\cdot u_{\Delta t}\right)^2}$$

$$= \pm\sqrt{\left(\frac{1}{\Delta t}\cdot u_{m_f}\right)^2 + \left(\frac{-1}{\Delta t}\cdot u_{m_e}\right)^2 + \left(\frac{m_e - m_f}{\Delta t^2}\cdot u_{\Delta t}\right)^2} = \pm\sqrt{\left(\frac{1}{(10s)}\cdot(2g)\right)^2 + \left(\frac{-1}{(10s)}\cdot(2g)\right)^2 + \left(\frac{(200g) - (400g)}{(10s)^2}\cdot(0.2s)\right)^2}$$

$$= \boxed{\pm 0.693\,\frac{g}{s}}$$

$$\dot{m} = \boxed{20.0 \pm 0.7\,\frac{g}{s}\ (95\%)} = 19.3\,\frac{g}{s} \rightarrow 20.7\,\frac{g}{s}\ (95\%)$$