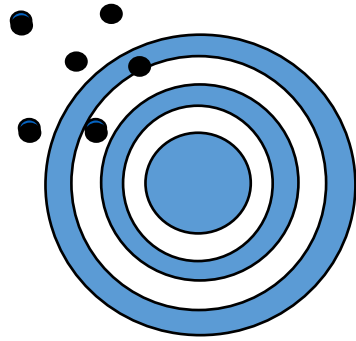


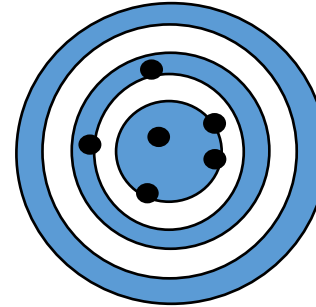
# Uncertainty Analysis

- **Types of Error**
- **Estimation of Uncertainty during**
  - **Design**
  - **Execution**
  - **Interpretation**

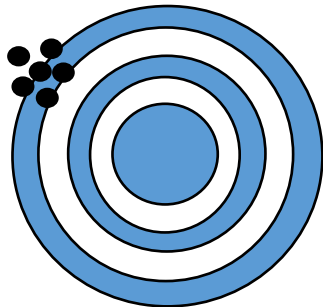
# Relating Accuracy to a combination of Precision and Bias



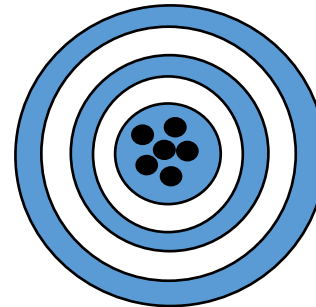
**Biased, Not Precise**



**Not Biased, Not Precise**



**Biased, Precise**



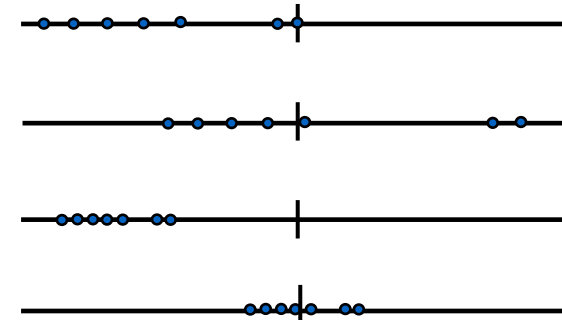
**Not Biased, Precise  
= ACCURATE**

# Types of Error

- Measured Value Components

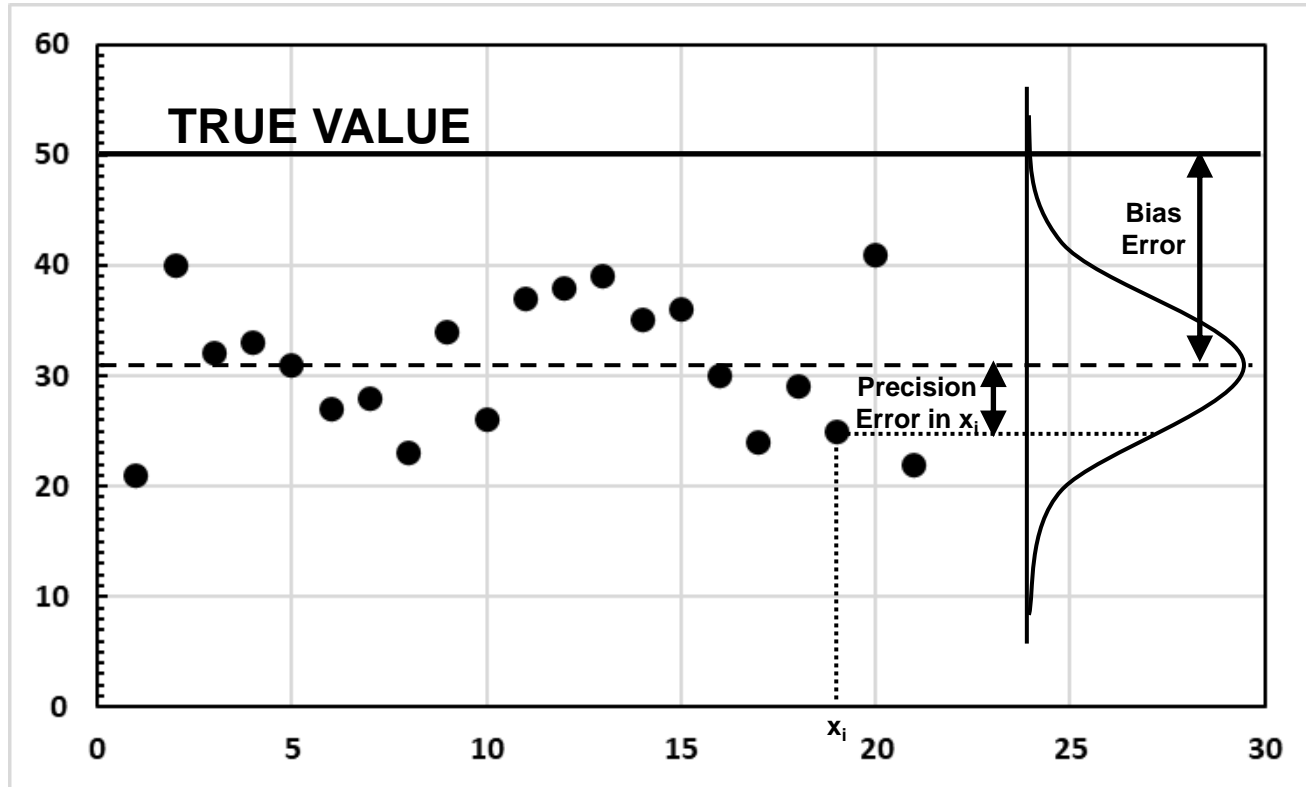
$$\text{Measured Value} = \text{True Value} + \text{Bias} + \text{Random Error}$$

$$\text{Measured Value} = \text{True Value} + (\mu - \text{True Value}) + \frac{\text{Precision}}{\sigma}$$



- Bias: Fixed or Systematic
  - Assumed to be small
  - Equipment Calibrated
- Precision (Uncertainty): Random Error
  - Accumulated
- ACCURACY deals with the difference between the *true* value and the *measured* value
  - An ACCURATE measurement has both small BIAS and PRECISION errors.

# Best Estimate of True Value, Sample Mean plus Uncertainty



$$x = \bar{x} \pm u_a$$

$$u_x \equiv \text{bias} + \text{uncertainty in } x$$

$$\bar{x} \equiv \text{sample mean}$$

$$x \equiv \text{true value estimate without bias}$$

# Error Estimates

If an error is statistically estimated, treat it as a **PRECISION** error; otherwise, treat it as a **BIAS** error

## BIAS

- Can not be discerned by statistical means alone
- Each measurement contains the same amount of Bias
- Bias can only be estimated by comparison
  - Calibration
  - Concomitant Methodologies
  - Interlaboratory comparisons
  - Experience

## PRECISION

- Can be statistically estimated
- Scatter in data generated under nominally fixed operating conditions
  - **Measurement System**
    - Repeatability and Resolution
  - **Measured Variable**
    - Temporal and Spatial Variations
  - **Process**
    - Variations in operating and environmental conditions
  - **Measurement Procedure and Technique**
    - Repeatability

# Standards

[illegible]

 <b>SAE</b> INTERNATIONAL	<b>AEROSPACE MATERIAL SPECIFICATION</b>	<b>AMS5356™</b>	<b>REV. J</b>
	Issued 1995-03 Revised 2008-10 Reaffirmed 2015-10 Superseding AMS5356H		
Steel, Corrosion Resistant, Investment Castings 16Cr - 4.1Ni - 0.28Cb - 3.2Cu Homogenization and Solution Heat Treated or Homogenization, Solution, and Precipitation Heat Treated (Composition similar to UNS J32200)			

**RATIONALE**

AMSS355J has been reaffirmed to comply with the SAE five-year review policy.

1. SCOPE

1.1 Form

This specification covers the corrosion resistant steel in the form of investment castings.

1.2 Application

These castings have been used typically for parts requiring good corrosion resistance and strength up to 600 °F (316 °C), but usage is not limited to such applications (See 3.3).

2.1 Certain processing procedures and service conditions may cause these castings to become subject to stress-corrosion cracking. ARP1110 recommends practices to minimize such conditions. Where stress-corrosion is considered to be a factor, pre-heat treatment should be performed at a temperature not lower than 1000 °F (538 °C).

2. APPLICABLE DOCUMENTS

The issue of the following documents is in effect on the date of the purchase order forms a part of this specification to the extent specified herein. The supplier may work to a subsequent revision of a document unless a specific document issue is specified. Where the referenced document has been cancelled and no superseding document has been specified, the last published issue of that document shall apply.


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**ASME Y14.24-1999**  
(Revision of ASME Y14.24M-1989 (R1993))

**REAFFIRMED 2004**  
FOR CURRENT COMMITTEE PERSONNEL  
PLEASE E-MAIL [CS@asme.org](mailto:CS@asme.org)

# TYPES AND APPLICATIONS OF ENGINEERING DRAWINGS

AN AMERICAN NATIONAL STANDARD

 The American Society of  
Mechanical Engineers

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Printed on acid-free paper. This standard is part of the ASME Y14.24-1999 (Revision of ASME Y14.24M-1989 (R1993)) series of standards, which is available in microfilm and microfiche editions. For more information, contact the American Society of Mechanical Engineers, 11 W. 41st St., New York, NY 10018-5900.

Designation: D 3039/D 3039M - 00

Standard Test Method for  
Tensile Properties of Polymer Matrix Composite Materials<sup>1</sup>

This standard is issued under the fixed designation D 3030/D 3030M; the number immediately following the designation indicates the year of original adoption or, in the case of revision, the year of last revision. A number in parentheses indicates the year of last revision. A superscript number (n) indicates an editorial change since the last revision or supersession.

This standard has been approved for use by agencies of the Department of Defense.

1. **Scope**
  - 1.1 This test method determines the in-place tensile properties of polymer matrix composite materials reinforced by high strength fibers. The composite materials are limited to continuous fiber or discontinuous fiber-reinforced composites. The test method is not applicable to composites with respect to the test direction.
  - 1.2 This test method *does not purport to address all of the safety hazards*, of fibers associated with its use. It is the responsibility of the user of this standard to establish appropriate safety and health practices and determine the applicability of regulatory limitations prior to use.
  - 1.3 This test method is not intended for on-board units and may be regarded separately as standard. Within the test, the test method does not determine the failure mechanism; each system can not expect exposures as standard. Therefore, this test method may not adequately assess the failure. Combining values of the two exposure tests result in nonconformance with the standard.
2. **Referenced Documents**
  - 2.1 ASTM Standards:
    - D 692 Test Method for Density and Specific Gravity (Relative Density) of Plastics by Displacement
    - D 883 Terminology Relating to Plastics
    - D 2584 Test Method for Ignition Loss of Cured Reinforced Plastics
    - D 2734 Test Method for Void Content of Reinforced Plastics
    - D 3117 Test Method for Constituent Content of Composites Materials
    - D 3178 Test Method for Composite Materials
    - D 5220/D 5298 Test Method for Moisture Absorption Characteristics and Equilibrium Conditions of Polymeric Materials
  - 2.2 Practices for Force Verification of Testing Machines<sup>1</sup>
  - 2.3 Terminology Relating to Mechanical Testing<sup>2</sup>
  - 2.4 Practice for Verification and Classification of Extrusion Die Castings<sup>3</sup>
  - 2.5 Test Method for Young's Modulus, Tangent Modulus, and Poisson's Ratio of Plastics<sup>4</sup>
  - 2.6 Practice for Choice of Sample Size to Estimate a Mean<sup>5</sup>
  - 2.7 Test Method for Penetration Rate and Room Temperature Impact<sup>6</sup>
  - 2.8 Practice for Use of the Terms Precision and Bias in ASTM Test Methods<sup>7</sup>
  - 2.9 Practice for Performance Characteristics of Metallic Bonded Resin Strain Gages<sup>8</sup>
  - 2.10 Practice for Conducting an Interlaboratory Study to Determine the Precision of a Test Method<sup>9</sup>
  - 2.11 Practice for Verifying Interstrain Alignment Under Tensile Loading<sup>10</sup>
  - 2.12 Guide for Installing Bonded Resistance Strain Gages<sup>11</sup>
3. **Terminology**
  - 3.1 **Definitions:**—Terminology D 3178 defines terms relating to composite materials. Terms and their composites. Terminology D 883 defines terms relating to plastics. Terminology E 484 defines terms relating to mechanical testing. Terminology E 486 and Practice E 171 define terms relating to statistics. In this test method, the terms "precision" and "bias" are used with the same meaning as in Terminology D 3178 shall have precedence over the other standards.
  - 3.2 **Definition of Terms:** Specify to This Standard:
    - 3.2.1 **Accuracy:**—The closeness of a physical quantity

<sup>1</sup> This test method is under the jurisdiction of ASTM Committee D-30 on Composite Materials and is the direct responsibility of Subcommittee D30.04 on Lap and Lap-Joint Test Methods.

<sup>2</sup> *Annual Book of ASTM Standards*, Vol 03.01

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# Uncertainty Analysis Assumptions

1. The test objectives are known.
2. The measurement is a clearly defined process in which all known CALIBRATION corrections for bias error have already been applied.
3. Data are obtained under fixed operating conditions
4. The engineers have some experience with the system components.
  - a. Personal experience through previous tests or simulations
  - b. Manufacturer's literature
  - c. NIST bulletins
  - d. Professional Society codes and [standards](#)
    - i. ASTM, ANSI, ASME, ASE, ISO, Etc.
  - e. Technical literature

# Stages of Uncertainty Analysis

## **1. Design Stage Uncertainty**

- a. An analysis of performance before the measurement**
- b. Selecting instruments**
- c. Selection measurement techniques**
- d. Selection based on performance and cost**

## 2. Advanced Stage and/or Single Measurement Uncertainty

## 3. Multiple Measurement Uncertainty analysis



# Contributions to Design Stage Uncertainty

- **Zero-Order Uncertainty,  $u_0$** 
  - Uncertainty caused by the data scatter that results when the instrument is measured
  - **Arbitrary Rule: one-half of the instrument's resolution with a probability of 95%**  
$$u_0 = \pm \frac{1}{2} \cdot \text{Resolution} \text{ (95\%)}$$
  - **1 TRUE value in 20 falls outside the  $u_0$  range**
- **Manufacturer's statement concerning error,  $u_c$** 
  - **Catalog value for the type of instrument under ideal conditions.**
- **Errors combined using root-sum-squares (RSS) method**

$$\begin{aligned} u_d &= \pm \sqrt{e_1^2 + e_2^2 + \cdots + e_k^2} \text{ (95\%)} \\ &= \pm \sqrt{u_0^2 + u_c^2} \text{ (95\%)} \end{aligned}$$

# Example 1: Force Measurement

Consider the force-measuring instrument described by the catalog data that follows. Provide an estimate of the uncertainty attributable to this instrument and the instrument design stage uncertainty.

Resolution:	0.25 N
Range:	0-100 N
Linearity:	within 0.20 N over the range
Repeatability:	within 0.30 over the range

# Solution 1: Force Measurement

Resolution: 0.25 N  
Range: 0-100 N  
Linearity: within 0.20 N over the range  
Repeatability: within 0.30 over the range

The interpolation error (1/2 Resolution)  $u_0$  is

$$u_o = 0.25N/2 = 0.125N \text{ (95\%)}$$

The catalog uncertainty is due to both Linearity and Repeatability

$$\begin{aligned} u_c &= \pm\sqrt{(0.2N)^2 + (0.3N)^2} \\ &= \pm 0.36N \text{ (95\%)} \end{aligned}$$

The Design Stage Uncertainty is

$$\begin{aligned} u_d &= \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{(0.125N)^2 + (0.36N)^2} \\ &= \pm 0.38N \text{ (95\%)} \end{aligned}$$

# Example 2: Voltmeter/Pressure Transducer

A voltmeter is to be used to measure the output from the pressure transducer that outputs an electrical signal. The normal pressure expected will be ~**3 psi**. Estimate the design-stage uncertainty in this combination. The following information is available.

Voltmeter:

**Resolution:**  $10 \mu\text{V}$

**Accuracy:** within 0.001% of reading

Transducer

**Range:**  $\pm 5 \text{ psi}$

**Sensitivity:**  $1 \text{ V/psi}$

**Input Power:**  $10 \text{ V}_{\text{dc}} \pm 1\%$

**Output:**  $\pm 5 \text{ V}$

**Linearity:** within 2.5 mV/psi over the range

**Repeatability:** within 2 mV/psi over the range

**Resolution:** Negligible

# Solution 2: Voltmeter/Pressure Voltmeter

A voltmeter is to be used to measure the output from the pressure transducer that outputs an electrical signal. The normal pressure expected will be ~3 psi. Estimate the design-stage uncertainty in this combination. The following information is available.

## Voltmeter:

Resolution: 10  $\mu$ V  
Accuracy: within 0.001% of reading

## Transducer:

Resolution: Negligible  
Range:  $\pm 5$  psi  
Sensitivity: 1 V/psi  
Input Power: 10 V<sub>dc</sub>  $\pm$  1%  
Output:  $\pm 5$  V  
Linearity: within 2.5 mV/psi over the range  
Repeatability: within 2 mV/psi over the range

Uncertainty of the **VOLTMETER:**  $u_d = \pm \sqrt{u_0^2 + u_c^2}$

Zeroth order Uncertainty:  $u_0 = 10\mu V / 2 = \boxed{\pm 5\mu V \quad (95\%)}$

For a nominal pressure of ~3 psi and a sensitivity of 1V/psi, expected output 3V

$$u_c = 3V \cdot \frac{0.001\%}{100\%} = \boxed{\pm 30\mu V \quad (95\%)}$$

Translating a %  
Into a decimal value

$$\begin{aligned} \text{VOLTMETER Design Stage Uncertainty: } u_{dv} &= \pm \sqrt{u_0^2 + u_c^2} = \pm \sqrt{(\pm 5\mu V)^2 + (\pm 30\mu V)^2} \\ &= \boxed{\pm 30.4\mu V = \pm 0.0304 \text{ mv} \quad (95\%)} \quad \textcircled{1} \end{aligned}$$

# Solution 2: Voltmeter/Pressure Transducer

A voltmeter is to be used to measure the output from the pressure transducer that outputs an electrical signal. The normal pressure expected will be **~3 psi**. Estimate the design-stage uncertainty in this combination. The following information is available.

## Voltmeter:

Resolution: 10  $\mu$ V  
Accuracy: within 0.001% of reading

## Transducer:

Resolution: Negligible  
Range:  $\pm 5$  psi  
**Sensitivity:** 1 V/psi  
Input Power: 10 V<sub>dc</sub>  $\pm 1\%$   
Output:  $\pm 5$  V  
**Linearity:** within 2.5 mV/psi over the range  
**Repeatability:** within 2 mV/psi over the range

## Uncertainty of the PRESSURE TRANSDUCER

$$u_d = \pm \sqrt{u_0^2 + u_c^2}$$

$$u_0 = 0 \text{ mV} \quad (95\%)$$

For a nominal pressure of ~3 psi, using the linearity and repeatability, the **PRESSURE TRANSDUCER** design level uncertainty is

$$u_{dT} = \pm \sqrt{u_0^2 + u_{\epsilon}^2} = \sqrt{(2.5 \text{ mV/psi} \cdot 3 \text{ psi})^2 + (2 \text{ mV/psi} \cdot 3 \text{ psi})^2} = \boxed{\pm 9.61 \text{ mV} \quad (95\%)} \quad \textcircled{2}$$

The **TOTAL Design Stage Uncertainty:** ① and ② combined

$$u_d = \pm \sqrt{u_{dV}^2 + u_{dT}^2} = \pm \sqrt{(\pm 0.0304 \text{ mV})_{dV}^2 + (\pm 9.61 \text{ mV})_{dT}^2} = \boxed{\pm 9.61 \text{ mV} \quad (95\%) = \pm 0.0096 \text{ psi} \quad (95\%)}$$

# Example 3: Tachometer

A tachometer has an analog display dial graduated in 5 revolutions per minute (rpm) increments. The user manual states an accuracy of 1% of reading. Estimate the uncertainty in the reading at 10, 500, and 5000 rpm.

# Solution 3: Tachometer

A tachometer has an analog display dial graduated in 5 revolutions per minute (rpm) increments. The user manual states an accuracy of 1% of reading. Estimate the uncertainty in the reading at 10, 500, and 5000 rpm.

Resolution: 5 rpm  
Accuracy: within  $\pm 1\%$  of reading

## Zeroth Order Uncertainty $u_0$ :

$$u_0 = \pm \frac{5 \text{ rpm}}{2} = \pm 2.5 \text{ rpm} \quad (95\%)$$

## Catalog Uncertainty $u_c$ :

The accuracy is stated to be within  $\pm 1\%$  of the reading. For readings at 10, 500, and 5000 rpm.

$$(u_c)_{10} = \pm 1\% \cdot 10 \text{ rpm} = 0.01 \cdot 10 \text{ rpm} = \pm 0.1 \text{ rpm}$$

$$(u_c)_{500} = \pm 1\% \cdot 500 \text{ rpm} = 0.01 \cdot 500 \text{ rpm} = \pm 5.0 \text{ rpm}$$

$$(u_c)_{5000} = \pm 1\% \cdot 5000 \text{ rpm} = 0.01 \cdot 5000 \text{ rpm} = \pm 50.0 \text{ rpm}$$

## Design Stage Uncertainty $u_d$ :

Combining the zeroth order and catalog uncertainties using RSS

$$(u_d)_{10} = \pm \sqrt{(u_0)^2 + (u_c)_{10}^2} = \sqrt{(\pm 2.5 \text{ rpm})^2 + (\pm 0.1 \text{ rpm})^2} = \boxed{\pm 2.5 \text{ rpm} \quad (95\%)}$$

$$(u_d)_{500} = \pm \sqrt{(u_0)^2 + (u_c)_{500}^2} = \sqrt{(\pm 2.5 \text{ rpm})^2 + (\pm 5.0 \text{ rpm})^2} = \boxed{\pm 5.6 \text{ rpm} \quad (95\%)}$$

$$(u_d)_{5000} = \pm \sqrt{(u_0)^2 + (u_c)_{5000}^2} = \sqrt{(\pm 2.5 \text{ rpm})^2 + (\pm 50.0 \text{ rpm})^2} = \boxed{\pm 50.0 \text{ rpm} \quad (95\%)}$$



# Example 4: Automobile

An automobile speedometer is graduated in 5-mph (8kph) increments and has an accuracy rated to be within  $\pm 4\%$ . Estimate the uncertainty in indicated speed at 60 mph (90 kph)

# Solution 4: Automobile

An automobile speedometer is graduated in 5-mph (8kph) increments and has an accuracy rated to be within  $\pm 4\%$ . Estimate the uncertainty in indicated speed at 60 mph (90 kph)

Resolution: 5-mph (8kph)

Accuracy: within  $\pm 4\%$

## Zeroth Order Uncertainty $u_0$ :

$$u_0 = \pm \frac{5 \text{ mph}}{2} = \pm 2.5 \text{ mph} \quad (95\%) = 4 \text{ kph} \quad (95\%)$$

## Catalog Uncertainty $u_c$ :

The accuracy is stated to be within  $\pm 4\%$  of the reading. The indicated speed here is 60 mph (90 kph).

$$u_c = \pm 4\% \cdot 60 \text{ mph} = 0.04 \cdot 60 \text{ mph} = \pm 2.4 \text{ mph} @ 60 \text{ mph} = \pm 3.6 \text{ kph} @ 90 \text{ kph}$$

## Design Stage Uncertainty $u_d$ :

Combining the two solutions using RSS

$$u_d = \pm \sqrt{(u_0)^2 + (u_c)^2} = \sqrt{(\pm 2.5 \text{ mph})^2 + (\pm 2.4 \text{ mph})^2} = \begin{matrix} \pm 3.5 \text{ mph} @ 60 \text{ mph} (95\%) \\ \pm 5.4 \text{ kph} @ 90 \text{ kph} (95\%) \end{matrix}$$

# Example 5: Temperature

A temperature measurement system is composed of a sensor and a readout device. The readout device has an accuracy of  $0.6^{\circ}\text{C}$  with a resolution of  $0.1^{\circ}\text{C}$ . The sensor has an off-the-self accuracy of  $0.5^{\circ}\text{C}$ . Estimate the design-stage uncertainty in the temperature indicated by this combination of sensor and readout.

# Solution 5: Temperature

A temperature measurement system is composed of a sensor and a readout device. The readout device has an accuracy of  $0.6^{\circ}\text{C}$  with a resolution of  $0.1^{\circ}\text{C}$ . The sensor has an off-the-self accuracy of  $0.5^{\circ}\text{C}$ . Estimate the design-stage uncertainty in the temperature indicated by this combination of sensor and readout.

## Temperature Sensor

Error Limit:  $\pm 0.5^{\circ}\text{C}$

## Readout Device

Resolution:  $0.1^{\circ}\text{C}$

Accuracy:  $0.6^{\circ}\text{C}$

## Zeroth Order Uncertainty $u_0$ :

Sensor

$$u_{0s} = \pm 0 \text{ (no readout)}$$

Readout

$$u_0 = \pm \frac{0.1^{\circ}\text{C}}{2} = \pm 0.05^{\circ}\text{C} \text{ (95\%)}$$

## Catalog Uncertainty $u_c$ :

Sensor

$$u_{cs} = \pm 0.5^{\circ}\text{C}$$

Readout

$$u_{cR} = \pm 0.6^{\circ}\text{C}$$

## Design Stage Uncertainty $u_d$ :

Combining all the uncertainties using RSS

$$\begin{aligned} u_d &= \pm \sqrt{(u_{0s})^2 + (u_{cs})^2 + (u_{0R})^2 + (u_{cR})^2} \\ &= \pm \sqrt{(\pm 0^{\circ}\text{C})^2 + (\pm 0.05^{\circ}\text{C})^2 + (\pm 0.05^{\circ}\text{C})^2 + (\pm 0.6^{\circ}\text{C})^2} = \boxed{\pm 0.78^{\circ}\text{C} \text{ (95\%)}} \end{aligned}$$

# Example 6: Pressure Readout

An equipment catalog boasts that a pressure transducer system comes in 3 ½ digit (e.g., 19.99) or 4 ½ digit (e.g., 19.999) display (the half digit is the leading 1, this digit can only display the 1). The 4 ½ digit model costs 50% more. Both units are otherwise identical. The specifications are as follows:

Linearity error: 0.15% FSO

Hysteresis error: 0.20% FSO

Repeatability error: 0.25% FSO

(FSO  $\equiv$  Full Scale Output). For an FSO of 20 kPa, select a readout based on appropriate uncertainty calculations.

# Solution 6: Pressure Readout

An equipment catalog boasts that a pressure transducer system comes in 3 ½ digit (e.g., 19.99) or 4 ½ digit (e.g., 19.999) display. The 4 ½ digit model costs 50% more. Both units are otherwise identical. The specifications are as follows:

Linearity error: 0.15% FSO  $\rightarrow u_1 = 0.0015 \cdot 20 \text{ kPa} = 0.03 \text{ kPa}$   
 Hysteresis error: 0.30% FSO  $\rightarrow u_2 = 0.002 \cdot 20 \text{ kPa} = 0.04 \text{ kPa}$   
 Repeatability error: 0.25% FSO  $\rightarrow u_3 = 0.0025 \cdot 20 \text{ kPa} = 0.05 \text{ kPa}$

(FSO  $\equiv$  Full Scale Output). For an FSO of 20 kPa, select a readout based on appropriate uncertainty calculations.

## Zeroth Order Uncertainty $u_0$ :

3 ½ Readout

$$u_{0 \text{ } 3^{1/2}} = 0.01 \text{ kPa} \quad (95\%)$$

4 ½ Readout

$$u_{0 \text{ } 4^{1/2}} = 0.001 \text{ kPa} \quad (95\%)$$

## Catalog Uncertainty $u_c$ : (Same for both readout types)

Linearity error: 0.15% FSO  $\rightarrow u_1 = 0.0015 \cdot 20 \text{ kPa} = 0.03 \text{ kPa}$   
 Hysteresis error: 0.30% FSO  $\rightarrow u_2 = 0.002 \cdot 20 \text{ kPa} = 0.04 \text{ kPa}$   
 Repeatability error: 0.25% FSO  $\rightarrow u_3 = 0.0025 \cdot 20 \text{ kPa} = 0.05 \text{ kPa}$

$$u_c = \pm\sqrt{(u_{c1})^2 + (u_{c2})^2 + (u_{c3})^2} = \pm\sqrt{(0.03 \text{ kPa})^2 + (0.04 \text{ kPa})^2 + (0.05 \text{ kPa})^2} = \pm 0.0707 \text{ kPa} \quad (95\%)$$

## Design Stage Uncertainty $u_d$ :

Combining all the uncertainties using RSS

3 ½ Readout

$$\begin{aligned} u_{d \text{ } 3^{1/2}} &= \pm\sqrt{(u_{0 \text{ } 3^{1/2}})^2 + (u_c)^2} \\ &= \pm\sqrt{(0.01 \text{ kPa})^2 + (0.0707 \text{ kPa})^2} \\ &= \pm 0.0714 \text{ kPa} \quad (95\%) \end{aligned}$$

4 ½ Readout

$$\begin{aligned} u_{d \text{ } 4^{1/2}} &= \pm\sqrt{(u_{0 \text{ } 4^{1/2}})^2 + (u_c)^2} \\ &= \pm\sqrt{(0.001 \text{ kPa})^2 + (0.0707 \text{ kPa})^2} \\ &= \pm 0.0707 \text{ kPa} \quad (95\%) \end{aligned}$$

**The two readouts have the SAME Design Level Uncertainty for all practical purposes.**

# Example 7: Crank Shaft

You are to measure the rotational speed of a crankshaft. The available angular velocity meter has a display with a resolution of 2 rpm (revolutions/minute). The meter specifications indicate the device's accuracy is 1.5% of the velocity reading. Find the design stage uncertainty for each velocity: a) 40 rpm, b) 400 rpm, and c) 4000 rpm.

# Example 8: A/D Converter

A temperature measurement system is comprised of a low-voltage temperature sensor, an anti-aliasing filter, and an A/D converter. The sensor's output voltage is linearly proportional to the temperature with a sensitivity of 10mV/°C. The operational amplifier gains the sensor voltage by a factor of 10.

The manufacturers of the instruments provided the following accuracies and information.

Temperature Sensor: 1%

Operational Amplifier: 2%

Anti-Aliasing Filter: 0.5%

A/D Converter: 1%: 12 bit; 10 V full scale resolution

Determine the zero-th order uncertainty and the design stage uncertainty.



# Solution 8: A/D Converter

A temperature measurement system is comprised of a low-voltage temperature sensor, an anti-aliasing filter, and an A/D converter. The sensor's output voltage is linearly proportional to the temperature with a sensitivity of 10mV/°C. The operational amplifier gains the sensor voltage by a factor of 10.

The manufacturers of the instruments provided the following accuracies and information.

Sensitivity:	10mV/°C
Temperature Sensor:	1%
Operational Amplifier:	2%
Anti-Aliasing Filter:	0.5%
A/D Converter:	1%: 12 bit; 10 V full scale resolution

## Zeroth Order Uncertainty:

Resolution of a 12 bit A/D converter:

- 12 bit Resolution = 1 part of  $2^{12} = 1$  part of 4096

$$Resolution = \frac{Full\ Scale\ Range}{2^{12}\ bits} = \frac{10V}{4096\ bits} = 2.44 \frac{mV}{bit}$$

$$u_0 = \frac{bit}{2} = \underbrace{\frac{bit \cdot 2.44 mV/bit}{2}}_{\text{Conversion bit to mV}} = 1.22\ mV \cdot \underbrace{\left(\frac{0.001V}{mV}\right)}_{\text{Conversion mV to V}} = 0.00122\ V \quad \textcircled{1}$$

- As a percentage of Full Scale

$$u_0\ \%_{FS} = Percent\ Full\ Scale\ Zeroth\ Order\ Uncertainty = 0.00122\ V \cdot \left(\frac{100\%}{10\ V}\right) = \boxed{0.0122\%} \quad \textcircled{2}$$

# Solution 8: A/D Converter

A temperature measurement system is comprised of a low-voltage temperature sensor, an anti-aliasing filter, and an A/D converter. The sensor's output voltage is linearly proportional to the temperature with a sensitivity of 10mV/°C. The operational amplifier gains the sensor voltage by a factor of 10.

The manufacturers of the instruments provided the following accuracies and information.

Sensitivity:	10mV/°C
Temperature Sensor:	1%
Operational Amplifier:	2%
Anti-Aliasing Filter:	0.5%
A/D Converter:	1%: 12 bit; 10 V full scale resolution

## Design Stage Uncertainty:

RSS Calculation in % Uncertainty:

$$\begin{aligned}u_{d\%} &\equiv \pm \textit{Design Stage Uncertainty} = \pm \sqrt{u_0^2 + u_{\text{sensor}}^2 + u_{\text{amplifier}}^2 + u_{\text{filter}}^2 + u_{\text{converter}}^2} \\&= \pm \sqrt{(0.0122\%)^2 + (1.0\%)^2 + (2.0\%)^2 + (0.5\%)^2 + (1.0\%)^2} = \boxed{\pm 2.5\%}\end{aligned}$$

$$u_d = \textit{Design Stage Uncertainty} = \pm u_{d\%} \cdot \left(\frac{10\text{ V}}{100\%}\right) = \pm 2.5\% \cdot \left(\frac{10\text{ V}}{100\%}\right) = \boxed{\pm 0.250\text{ V}}$$

# Stages of Uncertainty Analysis

1. Design Stage Uncertainty

**2. Advanced Stage and/or Single Measurement Uncertainty**

**3. Multiple Measurement Uncertainty analysis**

# Estimation of Uncertainty

- The goal is to estimate the ***uncertainty*** of experimental measurements and calculate results due to random error
- Procedure
  - Estimate the ***uncertainty interval***  $\pm 2\sigma$  for each measured quantity
  - State the ***confidence limit*** on each measurement
    - Random Measurement error is assumed Normally Distributed
  - Analyze the propagation of uncertainty into results calculated from experimental data

# Estimate the Measurement Uncertainty Interval

- Measured Variables:  $x_1, x_2, \dots$
- Estimate Random Error
  - $\pm n\sigma$  where  $n=1,2,3\dots$
  - $\pm 3\sigma$ : 99 percent of measured values
  - $\pm 2\sigma$ : 95 percent of measured values
  - $\pm \sigma$ : 68 percent of measured values
- Single Sample
  - $\pm$  half the smallest scale division (the least count) of the instrument

# Analyze the Propagation of Uncertainty in Calculations

- Measurements of independent variables in lab,  $x_1, x_2, \dots$
- The relative uncertainty of each independently measured quantity is estimated as  $u_i$
- The measurements are used to calculate the result,  $R$
- We wish to estimate how the errors in the  $x_i$ 's propagate into the calculation of  $R$

# Calculation of Uncertainty - DIRECT

## Kline-McClintock Second Power Relation

Every Regular Equation has an error equation.

Every Error Equation has one term for each measured quantity

$$R = R(x_1, x_2, \dots, x_n)$$

$$\delta R_i = \frac{\partial R}{\partial x_i} \cdot \delta x_i \quad \text{variation}$$

$$u_{R_i} = \frac{\partial R}{\partial x_i} \cdot u_{x_i}$$

$$u_{R_i} \equiv x'_i s \quad \text{contribution to the uncertainty of } R$$

$$u_{x_i} \equiv x'_i s \quad \text{uncertainty}$$

Kline, SJ and McClintock, FA, "Describing uncertainties in single sample experiments," Mechanical Engineering (75), 1953.

# Uncertainty Interval - DIRECT

- Single Variable

$$u_{R_i} = \frac{\partial R}{\partial x_i} \cdot u_{x_i}$$

- Multiple Variables

$$R = R(x_1, x_2, \dots, x_n)$$

$$\delta R_i = \frac{\partial R}{\partial x_i} \cdot \delta x_i \quad \text{variation}$$

$$u_{R_i} = \frac{\partial R}{\partial x_i} \cdot u_{x_i}$$

- Absolute Uncertainty

$$u_R = \pm \left[ \left( \frac{\partial R}{\partial x_1} \cdot u_{x_1} \right) + \left( \frac{\partial R}{\partial x_2} \cdot u_{x_2} \right) + \dots + \left( \frac{\partial R}{\partial x_i} \cdot u_{x_i} \right) + \dots + \left( \frac{\partial R}{\partial x_n} \cdot u_{x_n} \right) \right]$$

- Constant Odds Uncertainty – Accepted Method

$$u_R = \pm \left[ \left( \frac{\partial R}{\partial x_1} \cdot u_{x_1} \right)^2 + \left( \frac{\partial R}{\partial x_2} \cdot u_{x_2} \right)^2 + \dots + \left( \frac{\partial R}{\partial x_i} \cdot u_{x_i} \right)^2 + \dots + \left( \frac{\partial R}{\partial x_n} \cdot u_{x_n} \right)^2 \right]^{1/2}$$



# Uncertainty Interval - NORMALIZED

$$R = R(x_1, x_2, \dots, x_n)$$

$$\delta R_i = \frac{\partial R}{\partial x_i} \cdot \delta x_i \xrightarrow{\text{Multiply both sides by } 1/R} \frac{\delta R_i}{R} = \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{R}$$

$$\xrightarrow{\text{Multiply the Right-Hand side by 1 in the form } x_i/x_i} \frac{\delta R_i}{R} = \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{R} \cdot \frac{x_i}{x_i} = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{x_i}$$

$$\frac{\delta R_i}{R} \equiv \text{The normalized contribution to the uncertainty of } R \text{ by } x_i = U_i$$

$$\frac{\delta x_i}{x_i} \equiv \text{The normalized uncertainty of } x_i = U_{x_i}$$

$$\frac{\delta R_i}{R} = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot \frac{\delta x_i}{x_i} \longrightarrow U_i = \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot U_{x_i}$$

$$U_R \equiv \text{the normalized uncertainty of } R = \frac{\delta R}{R} = \pm \left[ \sum_{i=1}^n U_i^2 \right]^{1/2}$$

$$= \pm \left[ \left( \frac{x_1}{R} \cdot \frac{\partial R}{\partial x_1} \cdot U_{x_1} \right)^2 + \left( \frac{x_2}{R} \cdot \frac{\partial R}{\partial x_2} \cdot U_{x_2} \right)^2 + \dots + \left( \frac{x_i}{R} \cdot \frac{\partial R}{\partial x_i} \cdot U_{x_i} \right)^2 + \dots + \left( \frac{x_n}{R} \cdot \frac{\partial R}{\partial x_n} \cdot U_{x_n} \right)^2 \right]^{1/2}$$

Kline, SJ and McClintock, FA, "Describing uncertainties in single sample experiments," Mechanical Engineering (75), 1953.

# Example 9 : Volume Uncertainty, Analytical Approach

- Determine the Uncertainty in the Volume of Cylinder
  - $d = 2.9 \pm 0.16$  cm
  - $h = 1.5 \pm 0.11$  cm
  - $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

First, consider the extremes in the measurement of volume.

Next, consider both analytical approaches (Direct and Normalized)

# Solution 9 : DIRECT Approach

- Determine the Uncertainty in the Volume of Cylinder
  - $d = 2.9 \pm 0.16 \text{ cm}$
  - $h = 1.5 \pm 0.11 \text{ cm}$
  - $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

Extreme Value

	$-\Delta$	$+\Delta$
d	2.74 cm	3.06 cm
h	1.39 cm	1.61 cm

Analytical Approach – Direct

$$V = \frac{\pi}{4} \cdot d^2 \cdot h = \frac{\pi}{4} \cdot (2.9 \text{ cm})^2 \cdot (1.5 \text{ cm}) = 9.91 \text{ cm}^3$$

$$\begin{aligned}
 u_V &= \pm \left[ \left( \frac{\partial V}{\partial d} \cdot u_d \right)^2 + \left( \frac{\partial V}{\partial h} \cdot u_h \right)^2 \right]^{1/2} = \pm \left[ \left( \frac{\partial \left( \frac{\pi}{4} \cdot d^2 \cdot h \right)}{\partial d} \cdot u_d \right)^2 + \left( \frac{\partial \left( \frac{\pi}{4} \cdot d^2 \cdot h \right)}{\partial h} \cdot u_h \right)^2 \right]^{1/2} \\
 &= \pm \left[ \left( \frac{\pi}{4} \cdot 2 \cdot d \cdot h \cdot u_d \right)^2 + \left( \frac{\pi}{4} \cdot d^2 \cdot u_h \right)^2 \right]^{1/2} \\
 &= \pm \left[ \left( \frac{\pi}{2} \cdot 2.9 \text{ cm} \cdot 1.5 \text{ cm} \cdot 0.16 \text{ cm} \right)^2 + \left( \frac{\pi}{4} \cdot (2.9 \text{ cm})^2 \cdot 0.11 \text{ cm} \right)^2 \right]^{1/2}
 \end{aligned}$$

$$= \boxed{\pm 1.31 \text{ cm}^3 \text{ (95\%)} \implies V = 9.91 \pm 1.31 \text{ cm}^3 = 8.60 \text{ cm}^3 \rightarrow 11.22 \text{ cm}^3}$$

# Solution 9: Volume Uncertainty-Normalized

- Determine the Uncertainty in the Volume of Cylinder

- $d = 2.9 \pm 0.16 \text{ cm}$

- $h = 1.5 \pm 0.11 \text{ cm}$

- $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

$$V = \frac{\pi}{4} \cdot d^2 \cdot h = \frac{\pi}{4} \cdot (2.9 \text{ cm})^2 \cdot (1.5 \text{ cm}) = 9.91 \text{ cm}^3$$

$$\frac{\delta d}{d} = \frac{\pm 0.16 \text{ cm}}{2.9 \text{ cm}} = \pm 0.0552 = U_d$$

$$\frac{\delta h}{h} = \frac{\pm 0.11 \text{ cm}}{1.5 \text{ cm}} = \pm 0.0733 = U_h$$

$$\begin{aligned} \frac{u_V}{V} = U_V &= \pm \left[ \left( \frac{d}{V} \cdot \frac{\partial V}{\partial d} \cdot U_d \right)^2 + \left( \frac{h}{V} \cdot \frac{\partial V}{\partial h} \cdot U_h \right)^2 \right]^{1/2} \\ &= \pm \left[ \left( \frac{d}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\partial \left( \frac{\pi}{4} \cdot d^2 \cdot h \right)}{\partial d} \cdot U_d \right)^2 + \left( \frac{h}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\partial \left( \frac{\pi}{4} \cdot d^2 \cdot h \right)}{\partial h} \cdot U_h \right)^2 \right]^{1/2} \\ &= \pm \left[ \left( \frac{d}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\pi}{4} \cdot 2 \cdot d \cdot h \cdot U_d \right)^2 + \left( \frac{h}{\frac{\pi}{4} \cdot d^2 \cdot h} \cdot \frac{\pi}{4} \cdot d^2 \cdot U_h \right)^2 \right]^{1/2} = \pm [(2 \cdot U_d)^2 + (U_h)^2]^{1/2} \end{aligned}$$

Exactly the same as  
the previous result

$$= \pm [(0.1104)^2 + (0.0733)^2]^{1/2} = \pm 0.133 \quad \Rightarrow \quad u_V = V \cdot U_V = (9.91 \text{ cm}^3) \cdot (\pm 0.133) = \boxed{1.31 \text{ cm}^3 \text{ (95\%)}}$$

# If the Equation is Too Complex

## Sequential Numerical Perturbation

1. Based on measurements for the independent variables under some fixed operating condition, calculate a result  $R_0$  where  $R_0=f(x_1, x_2, \dots, x_N)$ . This value fixes the operating point for the numerical approximation.
2. Next, increase the independent variables by their respective uncertainties and recalculate the results based on each of these new values. Call these values  $R_i^+$ . That is
  - $R_1^+=f(x_1+u_{x1}, x_2, \dots, x_N)$
  - $R_2^+=f(x_1, x_2+u_{x2}, \dots, x_N)$
  - $R_N^+=f(x_1, x_2, \dots, x_N+u_{xN})$
3. Next, similarly, decrease the independent variables by their respective uncertainties and recalculate the results based on each of these new values. Call these values  $R_i^-$ .
4. Finally, evaluate the approximation of the uncertainty contributions from each variable.

$$\delta R_i = \frac{|\delta R_i^+| - |\delta R_i^-|}{2} \approx \theta_i \cdot u_i$$
$$u_R = \pm \left[ \sum_{i=1}^n (\delta R_i)^2 \right]^{1/2} \quad (P\%)$$

# Example 10: Volume Uncertainty, Numerical Approach

- Determine the Uncertainty in the Volume of Cylinder
  - $d = 29 \pm 0.16$  cm
  - $h = 1.5 \pm 0.11$  cm
- $V = V(d,h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

# Solution 10: Volume Uncertainty, Numerical Approach

- Determine the Uncertainty in the Volume of Cylinder

- $d = 2.9 \pm 0.16 \text{ cm}$

- $h = 1.5 \pm 0.11 \text{ cm}$

- $V = V(d, h) = \frac{1}{4} \cdot \pi \cdot d^2 \cdot h$

$$V = \frac{\pi}{4} \cdot d^2 \cdot h$$

$$V_d^+ = \frac{\pi}{4} \cdot (2.9 \text{ cm} + 0.16 \text{ cm})^2 \cdot (1.5 \text{ cm}) = 11.03 \text{ cm}^3$$

$$V_d^- = \frac{\pi}{4} \cdot (2.9 \text{ cm} - 0.16 \text{ cm})^2 \cdot (1.5 \text{ cm}) = 8.84 \text{ cm}^3$$

$$\delta V_d = \frac{|V_d^+| - |V_d^-|}{2} = \frac{|11.03 \text{ cm}^3| - |8.84 \text{ cm}^3|}{2} = 1.094 \text{ cm}^3$$

$$V_h^+ = \frac{\pi}{4} \cdot (2.9 \text{ cm})^2 \cdot (1.5 \text{ cm} + 0.11 \text{ cm}) = 10.63 \text{ cm}^3$$

$$V_h^- = \frac{\pi}{4} \cdot (2.9 \text{ cm})^2 \cdot (1.5 \text{ cm} - 0.11 \text{ cm}) = 9.18 \text{ cm}^3$$

$$\delta V_h = \frac{|V_h^+| - |V_h^-|}{2} = \frac{|10.63 \text{ cm}^3| - |9.18 \text{ cm}^3|}{2} = 0.73 \text{ cm}^3$$

The analytical and numerical approaches yield the same result. This will not always be the case.

Exactly the same as the previous results

$$u_V = \pm \left[ (1.094 \text{ cm}^3)^2 + (0.73 \text{ cm}^3)^2 \right]^{1/2} = \boxed{\pm 1.31 \text{ cm}^3 (95\%)}$$

# Example 11: Liquid Mass Flow Rate

The mass flow rate of water through a tube is determined by collecting water in a beaker. The mass flow rate is calculated from the net mass in the water collected divided by the time interval. Determine the uncertainty of the mass flow rate.

- Mass Flow Rate

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

$$\Delta m = m_f - m_e$$

- Error Estimates for measured Quantities
  - $M_f = 400 \pm 2\text{g}$  (20 to 1)
  - $M_e = 200 \pm 2\text{g}$  (20 to 1)
  - $\Delta t = 10 \pm 0.2\text{s}$  (20 to 1)



# Solution 11: Liquid Mass Flow Rate

- Mass Flow Rate
- Error Estimates for measured Quantities
  - $M_f = 400 \pm 2g$  (20 to 1)
  - $M_e = 200 \pm 2g$  (20 to 1)
  - $\Delta t = 10 \pm 0.2s$  (20 to 1)

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

$$\Delta m = m_f - m_e$$

$$\dot{m} = \frac{\Delta m}{\Delta t} = \frac{m_f - m_e}{\Delta t} = \frac{(400g) - (200g)}{(10s)} = 20 \frac{g}{s}$$

$$u = \pm \sqrt{\left(\frac{\partial \dot{m}}{\partial m_f} \cdot u_{m_f}\right)^2 + \left(\frac{\partial \dot{m}}{\partial m_e} \cdot u_{m_e}\right)^2 + \left(\frac{\partial \dot{m}}{\partial \Delta t} \cdot u_{\Delta t}\right)^2} = \sqrt{\left(\frac{\partial \left(\frac{m_f - m_e}{\Delta t}\right)}{\partial m_f} \cdot u_{m_f}\right)^2 + \left(\frac{\partial \left(\frac{m_f - m_e}{\Delta t}\right)}{\partial m_e} \cdot u_{m_e}\right)^2 + \left(\frac{\partial \left(\frac{m_f - m_e}{\Delta t}\right)}{\partial \Delta t} \cdot u_{\Delta t}\right)^2}$$

$$= \pm \sqrt{\left(\frac{1}{\Delta t} \cdot u_{m_f}\right)^2 + \left(\frac{-1}{\Delta t} \cdot u_{m_e}\right)^2 + \left(\frac{m_e - m_f}{\Delta t^2} \cdot u_{\Delta t}\right)^2} = \pm \sqrt{\left(\frac{1}{(10s)} \cdot (2g)\right)^2 + \left(\frac{-1}{(10s)} \cdot (2g)\right)^2 + \left(\frac{(200g) - (400g)}{(10s)^2} \cdot (0.2s)\right)^2}$$

$$= \boxed{\pm 0.693 \frac{g}{s}}$$

$$\dot{m} = \boxed{20.0 \pm 0.7 \frac{g}{s} \text{ (95\%)}} = 19.3 \frac{g}{s} \rightarrow 20.7 \frac{g}{s} \text{ (95\%)}$$